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Virasoro character identities and Artin L-functions.

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ABSTRACT

Some identities between unitary minimal Virasoro characters at levels $m = 3, 4, 5$ are shown to arise as a consequence of relations between Artin L-functions of different quadratic fields. The definitions and concepts of number theory necessary to present the theta function identities which can be derived from these relations are introduced. A new infinite family of identities between Virasoro characters at level 3 and level $m = 4a^2$, for a odd and $1 + 4a^2 = a'^2 p$ where p is prime is obtained as a by-product.

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1 Introduction

Two sets of intriguing identities between unitary minimal Virasoro characters were presented in [4]. They are remarkable in various respects. First they provide a rewriting of the Virasoro characters at level $m = 3$ (resp. $m = 4$) in terms of *differences* of quadratic expressions in Virasoro characters at level $m = 4$ (resp. $m = 3, 5$). Second, they are stronger identities than the ones obtained by repeated use of the Goddard-Kent-Olive sumrules [1] for the cosets $[SU(2)_1 \times SU(2)_1 \times SU(2)_1]/SU(2)_3$ and $[SU(2)_1 \times SU(2)_2 \times SU(2)_1]/SU(2)_4$. Third, their generalisation to higher levels is proving to be a highly nontrivial problem. The proof given in [4] uses infinite product representations of level $m = 3, 4, 5$ Virasoro characters, as well as the celebrated Jacobi triple identity and standard properties of generalised theta functions [2]. However, it does not shed any light on the underlying structure of these identities, and therefore offers no clue on how to generalise them to higher values of the level.

In this paper, we use the powerful machinery of number theory [3] to prove the same identities, and we provide a solid framework within which more identities can be unveiled. We establish relations between two imaginary quadratic extensions over $\mathbb{Q}$, which we call $\mathbf{K}$ and $\mathbf{K}'$ throughout. Given a Galois extension $\mathbf{N}$ of $\mathbb{Q}$ which contains $\mathbf{K}$ and $\mathbf{K}'$, we define two subgroups of the Galois group $\Gamma = \mathrm{Gal}(\mathbf{N}/\mathbb{Q})$ to be $\Delta = \mathrm{Gal}(\mathbf{N}/\mathbf{K})$ and $\Delta' = \mathrm{Gal}(\mathbf{N}/\mathbf{K}')$. The deep roots of the Virasoro identities considered lie in the ability to identify, given $\mathbf{N}$, pairs of characters χ_{gal} and χ'_{gal} of dimension one on Δ and Δ', which induce the *same* character on Γ,

$$\chi_{\mathrm{gal}\uparrow_\Delta^\Gamma} = {\chi'}_{\mathrm{gal}\uparrow_{\Delta'}^\Gamma}. \tag{1.1}$$

Now, by a standard result, the Artin L-function of an induced character coincides with the Artin L-function of the original character. So, given (1.1), one has,

$$L(\chi_{\mathrm{gal}}) = L(\chi_{\mathrm{gal}\uparrow_\Delta^\Gamma}) = L({\chi'}_{\mathrm{gal}\uparrow_{\Delta'}^\Gamma}) = L({\chi'}_{\mathrm{gal}}). \tag{1.2}$$

As explained later, the L-functions appearing in the expression (1.2) are related to ray class theta functions. The latter obey nontrivial identities which can be derived from the relations (1.2) as χ_{gal} and χ'_{gal} vary. Theorem 4.3 describes these identities and offers a practical way to identify relations between the generalised theta functions in terms of which the Virasoro characters can be expressed. However its formulation requires some ground work in number theory. We introduce the necessary mathematical jargon and results without the rather technical proofs, which will be presented elsewhere [5].

The paper is organised as follows. Section 2 sets the notations for coset theta functions [2], and emphasizes their rôle in the description of the Virasoro characters

[3] A standard textbook discussing the tools needed is [3]

identities considered in [4]. Since Theorem 4.3 is a result about ray class theta functions, we proceed in Section 3 to rewrite coset theta functions as ray class theta functions. In order to do so, we introduce ray class groups over $\mathbf{K}$ with conductor F. These provide a classification of ideals in $\mathcal{O}_{\mathbf{K}}$ (the ring of integers over $\mathbf{K}$) which generalises the classification modulo n of integers prime to n. The rôle of the modulus is played, in imaginary quadratic fields, by the conductor F. The three theorems at the end of Section 4 are the key to understanding the underlying algebraic structure of the Virasoro identities (2.4,2.5), but are also powerful tools in searching for more such identities. The first two subsections of Section 4 provide a brief discussion of the ray class characters and norm maps required in Theorems 4.2, 4.3 and 4.4. We show in Section 5 how the Virasoro identities presented in [4] arise as a consequence of Theorems 4.3 and 4.4 when $\mathbf{K} = \mathbb{Q}[\sqrt{-2}]$ (resp. $\mathbf{K} = \mathbb{Q}[\sqrt{-30}]$) and $\mathbf{K}' = \mathbb{Q}[i]$ (resp. $\mathbf{K}' = \mathbb{Q}[\sqrt{-10}]$). As an illustration of the power of the tools developed in this paper, we also provide an infinite family of identities between unitary minimal Virasoro characters at level $m = 3$ and unitary minimal Virasoro characters at level $m = 4a^2$ for a odd and $1 + 4a^2 = a'^2 p$ with p prime. The first member of this family, at $a = 1$, is the collection of identities (2.4).

2 Coset theta functions

The $\frac{m(m-1)}{2}$ independent unitary minimal Virasoro characters at level m ($m \geq 2, m \in \mathbb{N}$) are analytic functions of the variable $q = e^{2i\pi\tau}, \tau \in \mathbb{C}, \mathrm{Im}\tau \geq 0$ defined by,

$$\chi_{r,s}^{Vir\ (m)}(q) = \eta^{-1}(q)\left[\theta_{r(m+1)-sm,m(m+1)}(q) - \theta_{r(m+1)+sm,m(m+1)}(q)\right], \tag{2.1}$$

with the integers r and s in the following ranges,

$$r = 1, 2, \ldots, m-1; \qquad s = 1, \ldots, r.$$

The generalised theta functions at positive integer k are given by,

$$\theta_{\ell,k}(q) = \sum_{n\in\mathbb{Z}} q^{k\left(n+\frac{\ell}{2k}\right)^2} \tag{2.2}$$

for ℓ integer, and the Dedekind eta function $\eta(q)$ can be rewritten as the difference of two generalised theta functions at level 6,

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=0}^{\infty}\left(1 - q^{n+1}\right) = \theta_{1,6}(q) - \theta_{5,6}(q). \tag{2.3}$$

From (2.1) and (2.3), it is easy to see that the identities presented in [4], namely,

$$\chi_{1,i}^{Vir\ (3)}(q) = \epsilon_{ijk}(-1)^{j+k}\chi_{j,4}^{Vir\ (4)}(q)\ \chi_{k,2}^{Vir\ (4)}(q) \tag{2.4}$$

and

$$\begin{aligned}
\chi_{1,1}^{Vir\ (4)}(q) \pm \chi_{3,1}^{Vir\ (4)}(q) &= \left[\chi_{1,1}^{Vir\ (3)}(q) \pm \chi_{2,1}^{Vir\ (3)}(q)\right] \left[\chi_{2,2}^{Vir\ (5)}(q) \mp \chi_{3,2}^{Vir\ (5)}(q)\right], \\
\chi_{1,2}^{Vir\ (4)}(q) \pm \chi_{3,2}^{Vir\ (4)}(q) &= \left[\chi_{1,1}^{Vir\ (3)}(q) \pm \chi_{2,1}^{Vir\ (3)}(q)\right] \left[\chi_{1,2}^{Vir\ (5)}(q) \mp \chi_{4,2}^{Vir\ (5)}(q)\right], \\
\chi_{2,1}^{Vir\ (4)}(q) &= \chi_{2,2}^{Vir\ (3)}(q) \left[\chi_{2,1}^{Vir\ (5)}(q) - \chi_{3,1}^{Vir\ (5)}(q)\right], \\
\chi_{2,2}^{Vir\ (4)}(q) &= \chi_{2,2}^{Vir\ (3)}(q) \left[\chi_{1,1}^{Vir\ (5)}(q) - \chi_{4,1}^{Vir\ (5)}(q)\right], \qquad (2.5)
\end{aligned}$$

can be rewritten as identities involving quadratic expressions in generalised theta functions at different levels. For instance, (2.4) becomes,

$$V(1,2)V(4-3i,3) = \epsilon_{ijk}(-1)^{j+k}V(5j-16,4)V(5k-8,4) \qquad (2.6)$$

where we have introduced the V functions,

$$V(r,m) = \theta_{r,m(m+1)} - \theta_{r(2m+1),m(m+1)}, \qquad (2.7)$$

with the properties,

$$\begin{aligned}
V(r,m) &= V(-r,m) = V(r+2km(m+1),m), \qquad k \in \mathbb{Z}; \\
V(r,m) &= -V(r(2m+1),m). \qquad (2.8)
\end{aligned}$$

(Here, and in future, we will not express the variable q.) We note that,

$$\begin{aligned}
&\chi_{r,s}^{Vir(m)} = V(r(m+1)-sm,m)/\eta \quad \text{and} \quad \chi_{r,r}^{Vir(m)} = V(r,m)/\eta, \\
&\eta = \theta_{1,6} - \theta_{5,6} = V(1,2). \qquad (2.9)
\end{aligned}$$

We first remark that the generalised theta functions (2.2) are coset theta functions in the following sense.

Let X be a suitably sparse subset of a real or hermitian, positive definite, inner product space V. Choose $d \in \mathbb{R}^+$. We define,

$$\theta(X;d) = \sum_{x\in X} q^{\|x\|^2/d}, \qquad (2.10)$$

and we note that for a scalar α, we may define $\alpha X = \{\alpha x \mid x \in X\}$ and then,

$$\theta(\alpha X; \mid \alpha \mid^2 d) = \theta(X;d). \qquad (2.11)$$

When X is a coset $v + \Lambda$ of a lattice Λ in V, we shall call the function $\theta(X;d)$ a *coset theta function*. If Λ' is a lattice in another space V', and if $v' \in V'$, we find that $\Lambda \times \Lambda'$ is a lattice in $V \oplus V'$, and that $(v+\Lambda) \times (v'+\Lambda') = (v,v') + \Lambda \times \Lambda'$. We then get,

$$\theta((v,v') + \Lambda \times \Lambda'; d) = \theta(v+\Lambda;d)\theta(v'+\Lambda';d). \qquad (2.12)$$

From (2.2), (2.10) and (2.11), we see that,

$$\theta_{r,k} = \theta(\frac{r}{2k} + \mathbb{Z}; \frac{1}{k}) = \theta(\frac{r}{2\sqrt{k}} + \sqrt{k}\mathbb{Z}; 1),$$

i.e. the generalised theta function $\theta_{r,k}$ is a theta function for the coset $X = \frac{r}{2k} + \mathbb{Z}$ of the lattice $\mathbb{Z}$ in $V = \mathbb{R}$. Then we find, identifying $\mathbb{R} \oplus \mathbb{R}$ with $\mathbb{R} \oplus i\mathbb{R} = \mathbb{C}$,

$$\begin{aligned}
\theta_{r,k}\theta_{s,\ell} &\overset{(2.12)}{=} \theta((r/2\sqrt{k}) \pm i(s/2\sqrt{\ell}) + (\sqrt{k}\mathbb{Z} + i\sqrt{\ell}\mathbb{Z}); 1) \\
&\overset{(2.11)}{=} \theta((r\lambda\ell_0 \pm s\mu\sqrt{D}) + 2h\lambda\ell_0\mu\langle\mu k_0, \lambda\sqrt{D}\rangle_{gp}; 4k\ell\ell_0/h) \\
&\overset{\text{def}}{=} \theta(\alpha + J; d) \qquad (2.13)
\end{aligned}$$

where $k, \ell \in \mathbb{N}$, $h = \text{hcf }(k, \ell)$, $\bar{k} = k/h = \mu^2 k_0$ and $\bar{\ell} = \ell/h = \lambda^2\ell_0$ with k_0 and ℓ_0 square free as is $D = -k_0\ell_0$. So the product of two generalised theta functions $\theta_{r,k}\theta_{s,\ell}$ in $\mathbb{R}$ is a theta function for the coset $X = v + \Lambda \overset{\text{def}}{=} \alpha + J$ of a lattice

$$J = 2h\lambda\ell_0\mu\langle\mu k_0, \lambda\sqrt{D}\rangle_{gp} \qquad (2.14)$$

with

$$\langle\mu k_0, \lambda\sqrt{D}\rangle_{gp} = \{\mu k_0 n + \lambda\sqrt{D}m \mid n, m \in \mathbb{Z}\}$$

and

$$\alpha = r\lambda\ell_0 \pm s\mu\sqrt{D}.$$

The next step is to rewrite this coset theta function in terms of a ray class theta function in imaginary quadratic fields. In order to do so, we introduce in the next section the relevant definitions and properties of imaginary quadratic fields.

3 From coset to ray class theta functions

Let $\mathbf{K}$ and $\mathbf{K}_0$ be two fields and $\mathbf{K} \supset \mathbf{K}_0$. Then $\mathbf{K}$ is called an *extension field* of $\mathbf{K}_0$. The *degree of the extension*, noted $[\mathbf{K} : \mathbf{K}_0]$, is the dimension of $\mathbf{K}$ as a vector space over $\mathbf{K}_0$. In particular, if $[\mathbf{K} : \mathbf{K}_0] = 2$, the extension is a *quadratic* extension. In this paper, we work mainly with imaginary quadratic extensions of $\mathbb{Q}$ and we introduce the basic definitions and concepts as applied to them.

3.1 Ideals and prime factorization in imaginary quadratic fields.

Let D be a negative integer with no square factor other than 1, and put

$$\mathbf{K} = \mathbb{Q}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\},$$

an imaginary quadratic extension of $\mathbb{Q}$. We define $\mathcal{O}_{\mathbf{K}}$ to be the ring of all algebraic integers in $\mathbf{K}$. This means that if $D \equiv 2$ or $3(\text{mod}4)$,

$$\mathcal{O}_{\mathbf{K}} = \mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\} = \langle 1, \sqrt{D}\rangle_{gp}, \qquad (3.1)$$

and if $D \equiv 1 (\text{mod} 4)$,

$$\mathcal{O}_{\mathbf{K}} = \mathbb{Z}\left[\tfrac{1}{2}(1+\sqrt{D})\right] = \left\langle 1, \tfrac{1}{2}(1+\sqrt{D})\right\rangle_{gp}.$$

It is easy to see that every number in $\mathbf{K}$ is the ratio of two algebraic integers. A unit of $\mathcal{O}_{\mathbf{K}}$ is an element of $\mathcal{O}_{\mathbf{K}}$ whose reciprocal lies in $\mathcal{O}_{\mathbf{K}}$. The group of units of $\mathcal{O}_{\mathbf{K}}$ is denoted ${\mathcal{O}_{\mathbf{K}}}^{\times}$. We note in passing,

Proposition 3.1 *The units of $\mathcal{O}_{\mathbf{K}}$ are those roots of unity which lie in $\mathbf{K}$, vis. $\{\pm 1, \pm i\}$ if $D = -1$, the 6th roots of unity if $D = -3$ and $\{\pm 1\}$ otherwise.*

A *fractional ideal* I of $\mathcal{O}_{\mathbf{K}}$ generated by $\alpha_1, ..., \alpha_n \in \mathbf{K}$ is the set of $\mathcal{O}_{\mathbf{K}}$-linear combinations of the generators α_i:

$$I = (\alpha_1, ..., \alpha_n)_{\mathcal{O}_{\mathbf{K}}} \stackrel{\text{def}}{=} \{\sum_{i=1}^{n} \gamma_i \alpha_i \mid \gamma_i \in \mathcal{O}_{\mathbf{K}}\}.$$

In other words, I is a lattice spanning $\mathbf{K}$ which is closed under multiplication by elements of $\mathcal{O}_{\mathbf{K}}$. A fractional ideal $(\alpha)_{\mathcal{O}_{\mathbf{K}}} = \alpha\mathcal{O}_{\mathbf{K}}$, generated by a single $\alpha \in \mathbf{K} \setminus \{0\}$ is called a *principal ideal*. Note that

$$\alpha\mathcal{O}_{\mathbf{K}} = \beta\mathcal{O}_{\mathbf{K}} \iff \alpha/\beta \in {\mathcal{O}_{\mathbf{K}}}^{\times}. \tag{3.2}$$

We write $\mathcal{I}(\mathbf{K})$ for the set of all fractional ideals of $\mathcal{O}_{\mathbf{K}}$ and $\mathcal{P}(\mathbf{K})$ for the subset of those which are principal. $\mathcal{I}(\mathbf{K})$ is an abelian group under ideal multiplication, whose neutral element is $\mathcal{O}_{\mathbf{K}}$,

$$I.J = (\alpha_1, ...\alpha_n)_{\mathcal{O}_{\mathbf{K}}}.(\beta_1, ..., \beta_m)_{\mathcal{O}_{\mathbf{K}}} = \{\sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{ij}\alpha_i\beta_j \mid \gamma_{ij} \in \mathcal{O}_{\mathbf{K}}\}.$$

In particular, $\alpha\mathcal{O}_{\mathbf{K}}.\beta\mathcal{O}_{\mathbf{K}} = \alpha\beta\mathcal{O}_{\mathbf{K}}$, so $\mathcal{P}(\mathbf{K})$ is a subgroup of $\mathcal{I}(\mathbf{K})$. The *ideal class group*, $\mathcal{C}(\mathbf{K})$, is defined to be the quotient group of $\mathcal{I}(\mathbf{K})$ by $\mathcal{P}(\mathbf{K})$.

If $I \in \mathcal{I}(\mathbf{K})$ is a subset of $\mathcal{O}_{\mathbf{K}}$, then I is said to be an *integral ideal* of $\mathcal{O}_{\mathbf{K}}$. By a *prime ideal* of $\mathcal{O}_{\mathbf{K}}$ we shall mean an integral ideal which is contained in no other apart from $\mathcal{O}_{\mathbf{K}}$. (Technically, $\{0\}$ is also a prime ideal of $\mathcal{O}_{\mathbf{K}}$ but we shall ignore this fact.) Every fractional ideal $I \in \mathcal{I}(\mathbf{K})$ may be uniquely factorized as a product of integer powers of prime ideals Q_i,

$$I = \prod_{i=1}^{m} Q_i^{v_{Q_i}(I)}.$$

The integer $v_{Q_i}(I)$ is called the *valuation* of I at the prime ideal Q_i. Thus, only finitely many of the valuations $v_Q(I)$ are non-zero. As one might expect, $I \in \mathcal{I}(\mathbf{K})$ is integral if and only if $v_Q(I) \geq 0$ for all prime ideals Q. If I, $J \in \mathcal{I}(\mathbf{K})$, we say that I divides J (we write $I \mid J$) if JI^{-1} is an integral ideal. It follows that

$$v_Q(I) \leq v_Q(J)\ \forall Q \iff I \mid J \iff I \supset J.$$

Therefore the least common multiple of a pair of ideals I and J is the largest ideal contained in them both and their highest common factor is the smallest lattice which contains them both. Thus

$$\mathrm{lcm}(I,J) = I \cap J \quad \text{and} \quad \mathrm{hcf}(I,J) = I + J = \{a+b \mid a \in I, b \in J\}.$$

I is said to be *coprime* to J if their hcf is $\mathcal{O}_{\mathbf{K}}$. Note that in this case

$$I \cap J = \mathrm{lcm}(I,J) = IJ. \tag{3.3}$$

A useful parameter of integral ideals is the norm. If I is an integral ideal of $\mathcal{O}_{\mathbf{K}}$, we define its *norm* $\mathcal{N}(I)$ to be the (finite) number of elements in the quotient group $\mathcal{O}_{\mathbf{K}}/I$. The norm of a principal ideal $\alpha\mathcal{O}_{\mathbf{K}}$ is easily found since $\mathcal{N}(\alpha\mathcal{O}_{\mathbf{K}}) = |\alpha|^2$. Since any fractional ideal J is a quotient II'^{-1} of integral ideals, we may define $\mathcal{N}(J) = \mathcal{N}(I)/\mathcal{N}(I')$. This definition is unambiguous since, in fact, the norm is multiplicative:

$$\mathcal{N}(IJ) = \mathcal{N}(I)\mathcal{N}(J). \tag{3.4}$$

From this we see, also, that the norm of an integral ideal gives a clue as to its prime factors since (in a quadratic field) the norms of these must be p or p^2 for some prime number p.

Indeed, all prime ideals of $\mathbf{K}$ occur as prime factors of $p\mathcal{O}_{\mathbf{K}}$ for some prime number p. Either there is no ideal of norm p and $p\mathcal{O}_{\mathbf{K}}$ is itself prime or there is such an ideal, P_p. In this case,

$$p\mathcal{O}_{\mathbf{K}} = P_p\bar{P}_p, \tag{3.5}$$

and there are exactly one or two ideals of norm p according as $P_p = \bar{P}_p$ or not. In the rest of this paper P_p will stand for a prime ideal of $\mathcal{O}_{\mathbf{K}}$ of norm p (if there is one). We will make no explicit choice unless this is necessary.

If p is an odd prime integer, then the number of ideals of norm p in $\mathcal{O}_{\mathbf{K}}$ is the number of solutions to the congruence $x^2 \equiv D \bmod p$. (That is, $1 + \left(\frac{D}{p}\right)$ where $\left(\frac{D}{p}\right) = 0$ or ± 1 is the quadratic residue symbol, cf. subsection 4.1). (In all the cases that we will examine there will be just one ideal of norm 2.)

3.2 Ray class groups

Ray class theta functions are based on ray classes which we now introduce. These classify ideals in a way which generalises the classification, modulo n, of integers prime to n. For ideals of $\mathbf{K}$, the modulus n is replaced by an integral ideal F, called the *conductor*, and we work in the subgroup $\mathcal{I}_F(\mathbf{K})$ of quotients of those ideals which are prime to F, that is,

$$\mathcal{I}_F(\mathbf{K}) = \{I \in \mathcal{I}(\mathbf{K}) \mid v_P(I) = 0 \text{ if } P \mid F\}.$$

We identify a subgroup $\mathbf{K}_{1,F}$ of $\mathbf{K}^\times$ of elements which are, in some sense, 1 modulo F. These are quotients of elements of $\mathcal{O}_{\mathbf{K}}$ which are congruent mod F and coprime to F. Thus,

$$\mathbf{K}_{1,F} = \{\lambda/\mu \mid \lambda - \mu \in F,\ \mu\mathcal{O}_{\mathbf{K}} + F = \mathcal{O}_{\mathbf{K}}, \lambda\mu \neq 0\}. \tag{3.6}$$

We put $\mathcal{P}_F(\mathbf{K}) = \{\alpha\mathcal{O}_{\mathbf{K}} \mid \alpha \in \mathbf{K}_{1,F}\}$. Then the *ray class group* of $\mathcal{O}_{\mathbf{K}}$ with conductor F is the quotient group,

$$\mathcal{C}_F(\mathbf{K}) = \frac{\mathcal{I}_F(\mathbf{K})}{\mathcal{P}_F(\mathbf{K})}.$$

The ray class group is, in fact, a finite group. Note that, $\mathcal{C}_{\mathcal{O}_{\mathbf{K}}}(\mathbf{K}) = \mathcal{C}(\mathbf{K})$.

Each element of $\mathcal{C}_F(\mathbf{K})$ is a coset $I\mathcal{P}_F(\mathbf{K})$, for some $I \in \mathcal{I}_F(\mathbf{K})$. We denote this coset more compactly as $[I]_F$. We refer to it as the *ray class of I with conductor F.*

We shall work, in particular, with $\mathcal{CP}_F(\mathbf{K})$, the subgroup of $\mathcal{C}_F(\mathbf{K})$ of ray classes of principal ideals. We shall shorten our notation of such classes by writing $[\gamma]_F$ for $[\gamma\mathcal{O}_{\mathbf{K}}]_F$. Clearly, $\mathcal{CP}_F(\mathbf{K})$ is the kernel of the group homomorphism from $\mathcal{C}_F(\mathbf{K})$ to $\mathcal{C}(\mathbf{K})$ given by $[I]_F \mapsto [I]_{\mathcal{O}_{\mathbf{K}}}$. This homomorphism is, in fact, surjective and so we have a short exact sequence of groups,

$$0 \to \mathcal{CP}_F(\mathbf{K}) \to \mathcal{C}_F(\mathbf{K}) \overset{\mathrm{red}^F_{\mathcal{O}_{\mathbf{K}}}}{\to} \mathcal{C}(\mathbf{K}) \to 0. \tag{3.7}$$

We conclude with two useful lemmas (with F integral, as above).

Lemma 3.2 *Let $H \in \mathcal{I}(\mathbf{K})$. (i) If $\alpha\mathcal{O}_{\mathbf{K}} + FH = H$ then $\alpha H^{-1} \in \mathcal{I}_F(\mathbf{K})$.*

(ii) If, further, $\beta - \alpha \in FH$ then $\beta/\alpha \in \mathbf{K}_{1,F}$ and so $[\beta H^{-1}]_F = [\alpha H^{-1}]_F$.

(iii) Conversely, if α and $\beta \in H$ and $\beta/\alpha \in \mathbf{K}_{1,F}$ then $\beta - \alpha \in FH$.

Proof: (i) $\mathrm{hcf}(\alpha H^{-1}, F) = \alpha H^{-1} + F = (\alpha\mathcal{O}_{\mathbf{K}} + FH)H^{-1} = \mathcal{O}_{\mathbf{K}}$.

(ii) Since $\mathcal{C}(\mathbf{K})$ is finite, $H^n = \gamma\mathcal{O}_{\mathbf{K}}$ for some positive integer n and $\gamma \in \mathbf{K}$.

Put $\lambda = \beta\alpha^{n-1}/\gamma$ and $\mu = \alpha^n/\gamma$. Since $\alpha \in H$ and $\beta \equiv \alpha$ modulo FH, we have

$$\lambda \equiv \beta\alpha^{n-1}/\gamma \equiv \alpha^n/\gamma \equiv \mu \text{ modulo } FH^n/\gamma = F.$$

Moreover, $\mu\mathcal{O}_{\mathbf{K}} + F = (\alpha H^{-1})^n + F = \mathcal{O}_{\mathbf{K}}$. So $\beta/\alpha = \lambda/\mu \in \mathbf{K}_{1,F}$.

(iii) Now $\beta = \alpha(\lambda/\mu)\mathcal{O}_{\mathbf{K}}$, where $\lambda - \mu \in F$ and $\mu\mathcal{O}_{\mathbf{K}} + F = \mathcal{O}_{\mathbf{K}}$. Whence, by (3.3), $\mu\mathcal{O}_{\mathbf{K}} \cap F = \mu F$.

But $\beta - \alpha \in H$ and also $\beta - \alpha = (\lambda - \mu)\alpha/\mu \in FH(1/\mu)$. So

$$\beta - \alpha \in H \cap FH(1/\mu) = (1/\mu)H(\mu\mathcal{O}_{\mathbf{K}} \cap F) = (1/\mu)H(\mu F) = FH.$$

Lemma 3.3 *Suppose that α_1, α_2, α_3 and $\alpha_4 \in \mathbf{K} \setminus FH$ and $\alpha_i\mathcal{O}_{\mathbf{K}} + FH = H$. Then*

(i) $\alpha_i/\alpha_j = (\alpha_i H^{-1})(\alpha_j H^{-1})^{-1} \in \mathcal{I}_F(\mathbf{K})$.

And if $\alpha_1\alpha_2 \equiv \alpha_3\alpha_4$ modulo FH^2 then

(ii) $[\alpha_1/\alpha_4]_F = [\alpha_3/\alpha_2]_F$.

Proof: (i) is clear from Lemma 3.2(i). From Lemma 3.2(ii) — with H^2 instead of H —

$$[\alpha_1\alpha_2 H^{-2}]_F = [\alpha_3\alpha_4 H^{-2}]_F$$

and (ii) follows from this.

3.3 The ray class theta function.

For any ray class x in the ray class group $\mathcal{C}_F(\mathbf{K})$, we define the *ray class theta function* of x with scale factor $d \in \mathbb{R}^+$ as follows,

$$\theta(x; d) = \sum_{I \in x, I \subset \mathcal{O}_{\mathbf{K}}} q^{\mathcal{N}(I)/d}.$$

Here $\mathcal{N}(I)$ is the norm of I (see (3.4)). More generally, if $W = \sum_{x \in \mathcal{C}_F(\mathbf{K})} n_x x$ is a formal complex linear combination of ray classes, we write,

$$\theta(W; d) = \sum_{x \in \mathcal{C}_F(\mathbf{K})} n_x \theta(x; d). \tag{3.8}$$

Also, abusively, if X and Y are subsets of $\mathcal{C}_F(\mathbf{K})$, we can regard them as standing for the sums of their elements so that, for instance,

$$\theta(X - Y; d) = \sum_{x \in X} \theta(x; d) - \sum_{y \in Y} \theta(y; d).$$

Now the quadratic field coset theta functions as in (2.13) can be rewritten as ray class theta functions in the following way. (We do not tackle here the most general cases as we assume that the lattice J of (2.14) is an ideal of $\mathcal{O}_{\mathbf{K}}$. In the general case the coset must first be split up as a union of cosets of ideals (c.f. 5.19) and a sum of ray class theta functions will be obtained.)

Let $J \in \mathcal{I}(\mathbf{K})$ and $\alpha \in \mathbf{K} \setminus J$. Put $H = \mathrm{hcf}(\alpha\mathcal{O}_{\mathbf{K}}, J) = \alpha\mathcal{O}_{\mathbf{K}} + J$ and $F = JH^{-1}$, an integral ideal of $\mathcal{O}_{\mathbf{K}}$. Thus, by Lemma 3.2(i), $[\alpha H^{-1}]_F \in \mathcal{C}_F(\mathbf{K})$

Let $(\mathcal{O}_{\mathbf{K}}{}^\times)_F$ be the group of those units of $\mathcal{O}_{\mathbf{K}}$ which are 1 modulo F and let w_F be its order. We have the following relation between coset and ray class theta functions,

Theorem 3.4 *With α, J, F, and H as above.*

$$\theta(\alpha + J; d) = w_F \theta([\alpha H^{-1}]_F; d/\mathcal{N}(H)), \tag{3.9}$$

Proof: Put Y for the set of all integral ideals in $[\alpha H^{-1}]_F$.

The proof of the theorem is effected by the following lemma which sets up a w_F to 1 correspondence between the terms in the two theta sums and in which, since $\mathcal{N}(\beta H^{-1}) = |\beta|^2/\mathcal{N}(H)$, the powers of q are scaled by $1/\mathcal{N}(H)$.

Lemma 3.5 *The map $f : \alpha + J \to Y$ defined by $f(\beta) = \beta H^{-1}$ is w_F to 1 and surjective.*

Proof: By Lemma 3.2(ii), if $\beta \in \alpha + J$ then $f(\beta) \in [\alpha H^{-1}]_F$. But $\beta \in \alpha\mathcal{O}_{\mathbf{K}} + J = H$. So $f(\beta) \subset \mathcal{O}_{\mathbf{K}}$. Thus $f(\beta) \in Y$ and f is well-defined.

Let $I \in Y$. Then $I(\alpha H^{-1})^{-1} \in \mathcal{P}_F(\mathbf{K})$. So, $IH\alpha^{-1} = \delta\mathcal{O}_{\mathbf{K}}$, where $\delta \in \mathbf{K}_{1,F}$. Thus $IH = \beta\mathcal{O}_{\mathbf{K}}$ with $\beta = \alpha\delta$.

Now $\alpha \in H$ and $\beta \in IH \subset H$. So, by Lemma 3.2(iii), $\beta \in \alpha + FH = \alpha + J$. Of course, $f(\beta) = \beta H^{-1} = IHH^{-1} = I$. Therefore f is surjective.

To complete the proof we show that

$$f^{-1}(I) = \beta(\mathcal{O}_{\mathbf{K}}{}^{\times})_F.$$

Suppose $\omega \in (\mathcal{O}_{\mathbf{K}}{}^{\times})_F$. Then $\omega \in 1+F$ and so, since $\beta F \subset HF = J$, $\beta\omega \in \beta+J = \alpha+J$. Moreover, by (3.2), $f(\omega\beta) = f(\beta) = I$. Thus $f^{-1}(I)$ contains $\beta(\mathcal{O}_{\mathbf{K}}{}^{\times})_F$.

We must show the reverse inclusion. If $\beta' \in f^{-1}(I)$ then, certainly, $\beta'\mathcal{O}_{\mathbf{K}} = \beta\mathcal{O}_{\mathbf{K}}$ so, by (3.2), $\beta' = \omega\beta$ with $\omega \in \mathcal{O}_{\mathbf{K}}{}^{\times}$. But, by Lemma 3.2(ii) with $\beta' = \alpha$, $\omega = \beta'/\beta \in \mathbf{K}_{1,F}$. Thus, applying Lemma 3.2(iii) with $H = \mathcal{O}_{\mathbf{K}}$, $\omega \equiv 1$ modulo F. Thus $\beta' \in \beta(\mathcal{O}_{\mathbf{K}}{}^{\times})_F$, as required.

3.4 An example of reduction to ray classes

As an illustration of use of the above identity and of (2.13), and as a first step in proving the identities (2.4), we reduce a generic theta product occurring in the L.H.S. of (2.4) to a coset theta functions and then (in the main case) to a ray class theta function.

First, by (2.13),

$$\theta_{(1+6a),6}\theta_{s(1+6b),12} = \theta(\alpha_s(a,b) + J; d) \tag{3.10}$$

where (since, taking $k = 6$ and $\ell = 12$ we have $h = (6, 12) = 6$, $\lambda = \mu = k_0 = 1$ and $D = -2$) $\alpha \equiv \alpha_s(a, b) = 2(1 + 6a) + s(1 + 6b)\sqrt{D}$, $J = 24\langle 1, \sqrt{D}\rangle$ and $d = 96$.

We note that for a, b taking the values 0 or 1, and for $s = 1, -2, -5$, the L.H.S. of (3.10) gives all the terms in the three products $V(1, 2)V(4 - 3i, 3), i = 1, 2, 3$, which are the left hand sides of (2.4). We recall that generalised theta functions satisfy the following relations,

$$\theta_{\ell,k} = \theta_{-\ell,k} = \theta_{\ell+2k,k}.$$

So we work in the imaginary quadratic field $\mathbf{K} = \mathbb{Q}(\sqrt{-2})$ and we introduce the notation $\rho = \sqrt{-2}$. Then, from (3.1), $\mathcal{O}_{\mathbf{K}} = \langle 1, \rho\rangle$ and $J = 24\mathcal{O}_{\mathbf{K}}$. We find (see 3.5) prime factorisations $2\mathcal{O}_{\mathbf{K}} = P_2^2$ and $3\mathcal{O}_{\mathbf{K}} = P_3\bar{P}_3$, where $P_2 = \rho\mathcal{O}_{\mathbf{K}}$ and $P_3 = (1+\rho)_{\mathcal{O}_{\mathbf{K}}}$ with norms 2 and 3, respectively. So $J = P_2^6 P_3\bar{P}_3$.

In order to apply (3.9), we first need to determine the ideal H, which is the hcf of J and the principal ideal generated by $\alpha_s(a, b)$. Suppose that $s \equiv 1 \bmod 3$ and assume for the moment that s is odd (so $s \equiv 1 \bmod 6$). Then $\alpha = 2 + \rho + 6\beta$ for some $\beta \in \mathcal{O}_{\mathbf{K}}$ and $2 + \rho = \rho(1 - \rho)$. So

$$\alpha\mathcal{O}_{\mathbf{K}} = P_2\bar{P}_3(1 - \rho(1 + \rho)\beta)_{\mathcal{O}_{\mathbf{K}}},$$

where the last ideal is clearly not contained in P_2 or P_3. Thus $H = P_2\bar{P}_3$ and, using (3.4), the norm of H is,

$$\mathcal{N}(H) = \mathcal{N}(P_2)\mathcal{N}(\bar{P}_3) = 2 \times 3 = 6.$$

The conductor (i.e. F of (3.9)) is

$$JH^{-1} = P_2^6 P_3\bar{P}_3(P_2\bar{P}_3)^{-1} = P_2^5 P_3 = 4P_2P_3.$$

Thus, by (3.9), the coset theta function (3.10) reduces, for $s \equiv 1 \bmod 6$, to the ray class theta function,

$$\theta_{(1+6a),6}\theta_{s(1+6b),12} = \theta\left(\left[\frac{\alpha_s(a,b)}{\rho(1-\rho)}\right]_{4P_2P_3}; 16\right).$$

3.5 Description of principal ray classes and change of conductor.

Let $\mathbf{K}$ and F be again as in subsection 3.3. A standard result (though not difficult to prove) is the following description of the units of the quotient ring $\mathcal{O}_{\mathbf{K}}/F$:

$$(\mathcal{O}_{\mathbf{K}}/F)^\times = \{\alpha + F \mid \alpha\mathcal{O}_{\mathbf{K}} + F = \mathcal{O}_{\mathbf{K}}\}.$$

Combining Lemma 3.2 (with $H = \mathcal{O}_{\mathbf{K}}$) with (3.2), one can obtain the following exact sequence describing $\mathcal{CP}_F(\mathbf{K})$,

$$0 \to (\mathcal{O}_{\mathbf{K}}{}^\times)_F \to \mathcal{O}_{\mathbf{K}}{}^\times \to (\mathcal{O}_{\mathbf{K}}/F)^\times \stackrel{\pi_F}{\to} \mathcal{CP}_F(\mathbf{K}) \to 0 \tag{3.11}$$

Here the first map is inclusion, the second takes u to $u+F$ and π_F takes $\alpha+F$ to $[\alpha]_F$.

In our analysis of section 5, we will need to relate ray class groups corresponding to two conductors $\tilde{F}$ and F such that $\tilde{F} \mid F$. This relation is provided by the reduction map,

$$\mathrm{red}^F_{\tilde{F}} : \mathcal{C}_F(\mathbf{K}) \to \mathcal{C}_{\tilde{F}}(\mathbf{K}) : [I]_F \to [I]_{\tilde{F}}. \tag{3.12}$$

This is, in fact, a *surjective* group homomorphism of which the projection of (3.7) is a special case.

Comparing (3.7) for F and $\tilde{F}$ it is easy to see that the kernel of the reduction map $\mathrm{red}^F_{\tilde{F}}$ of (3.12) lies in $\mathcal{CP}_F(\mathbf{K})$. From (3.11) it now follows that

$$\ker \mathrm{red}^F_{\tilde{F}} = \pi_F\{\alpha + F \in (\mathcal{O}_{\mathbf{K}}/F)^\times \mid \alpha \equiv 1 \bmod \tilde{F}\}.$$

If $F = Q\tilde{F}$, and Q and $\tilde{F}$ have no common factors, one gets an isomorphism (the Chinese Remainder Theorem),

$$(\mathcal{O}_{\mathbf{K}}/F)^\times \to (\mathcal{O}_{\mathbf{K}}/Q)^\times \times (\mathcal{O}_{\mathbf{K}}/\tilde{F})^\times : \alpha + F \to (\alpha + Q, \alpha + \tilde{F}). \tag{3.13}$$

Clearly, isomorphisms of this sort are useful in the description of $\mathcal{CP}_F(\mathbf{K})$ and we generalise the compact notation introduced above for classes of principal ideals in the following way. If $\gamma + F$ is the pre-image of $(\alpha + Q, \beta + \tilde{F})$, we write $[\alpha, \beta]_F$ for $[\gamma]_F$. Note that for $u \in (\mathcal{O}_{\mathbf{K}})^\times$,

$$[\alpha, \beta]_F = [u\alpha, u\beta]_F. \tag{3.14}$$

It is easy to see that the kernel of $\mathrm{red}^F_{\tilde{F}}$ is $\{[\alpha,1]_F \mid \alpha+Q \in (\mathcal{O}_{\mathbf{K}}/Q)^\times\}$. More generally, if $A \subset \mathcal{CP}_{\tilde{F}}(\mathbf{K})$, then

$$(\mathrm{red}^F_{\tilde{F}})^{-1}(A) = \{[\alpha,\beta]_{Q\tilde{F}} \mid \alpha + Q \in (\mathcal{O}_{\mathbf{K}}/Q)^\times, \beta + \tilde{F} \in A\}. \tag{3.15}$$

If $\mathcal{N}(Q) = p$ then $\mathbb{Z}/p\mathbb{Z}$ and $\mathcal{O}_{\mathbf{K}}/Q$ have the same number p of elements. In fact, since $Q\bar{Q} = p\mathcal{O}_{\mathbf{K}}$ (see 3.5)), $\mathbb{Z} \cap Q = p\mathbb{Z}$ and so the map,

$$\mathbb{Z}/p\mathbb{Z} \to \mathcal{O}_{\mathbf{K}}/P_p \text{ by } n + p\mathbb{Z} \mapsto n + P_p, \tag{3.16}$$

is a ring isomorphism. Thus in this case the elements of $\mathcal{CP}_F(\mathbf{K})$ may be written $[n,\beta]_F$ with n an integer mod p.

4 Relations between ray class theta functions of different quadratic fields

The identities (2.4) and (2.5) are a consequence of relations between ray class theta functions of two different imaginary quadratic fields. However, to describe these relations, which are stated in Theorem 4.3, we need to define ray class characters for these two quadratic fields and their corresponding L-functions. We also use the theory of Artin L-functions in a crucial step (Theorem (4.2)).

4.1 Ray class characters

Let $\chi : \mathcal{C}_F(\mathbf{K}) \to \mathbb{C}^\times$ be a multiplicative character of the ray class group with conductor F. We define the *conductor* F_χ of the character χ to be the biggest of all ideals I dividing F such that the kernel of χ contains the kernel of the reduction map (3.12) from $\mathcal{C}_F(\mathbf{K})$ to $\mathcal{C}_I(\mathbf{K})$. In fact, F_χ is the sum of all such ideals. Since the reduction map is surjective, χ defines a character χ_{F_χ} on $\mathcal{C}_{F_\chi}(\mathbf{K})$ defined by the equation,

$$\chi([J]_F) = \chi_{F\chi}([J]_{F\chi}).$$

Thus, for all I divisible by $F\chi$, χ also defines a character χ_I of $\mathcal{C}_I(\mathbf{K})$ by,

$$\chi_I([J]_I) = \chi_{F\chi}([I]_{F\chi}).$$

Note that χ_I also has conductor $F\chi$. We refer to the collection of all such characters χ_I as "*the ray class character* χ *for* $\mathbf{K}$".

We are now in a position to specialise to the problem at hand. Let us introduce two complex quadratic fields $\mathbf{K} = \mathbb{Q}[\sqrt{D}]$ and $\mathbf{K}' = \mathbb{Q}[\sqrt{D'}]$ where D and D' are negative, square free integers. We shall also sometimes consider $D'' = DD'/(D,D')^2$, which is square free and positive, along with the quadratic field $\mathbf{K}'' = \mathbb{Q}[\sqrt{D''}] = \mathbb{Q}[\sqrt{DD'}]$. Here, (a,b) is the highest common factor of the pair of integers a, b.

The integer $\widetilde{D}$ is defined to be D or $4D$ according to whether 4 divides $(D-1)$ or not. The integers $\widetilde{D}'$ and $\widetilde{D}''$ are defined similarly. (These are the discriminants of $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{K}''$ over $\mathbb{Q}$.)

In order to define a particular ray class character ψ for $\mathbf{K}$, we first describe a function ϕ, which is, effectively, the Dirichlet character corresponding to the quadratic field $\mathbf{K}$.

Let p be a prime number and m an integer not divisible by p. Then the quadratic residue symbol, $\left(\frac{m}{p}\right)$ is defined to be 1 or -1 according to whether $x^2 \equiv m$ has a solution modulo p or not.

For positive integers n prime to $2D$ we define $\phi(n) = \pm 1$ by the following rules:

(i) $\phi(n)$ is the quadratic residue symbol $\left(\frac{D}{n}\right)$ if n is prime.

(ii) $\phi(n) = \phi(p_1)\phi(p_2)...\phi(p_r)$ if the prime factorization of n is $n = p_1p_2...p_r$.

(In fact $\phi(n)$ depends only on the residue of n modulo $\widetilde{D}$).

The function ϕ' is defined similarly to ϕ (using D' instead of D).

Let $F = 2DD'\mathcal{O}_{\mathbf{K}}$. We define the character ψ_F on $\mathcal{C}_F(\mathbf{K})$ by

$$\psi_F([I]_F) \stackrel{\text{def}}{=} \phi'(\mathcal{N}(I)),$$

where I is integral and $\mathcal{N}(I)$ is the norm of the ideal I — defined in the previous section before (3.4). Hence we obtain a ray class character ψ for $\mathbf{K}$. (This is the character corresponding the field extension $\mathbf{KK}'$ of $\mathbf{K}$.) The conductor of this character is (as shown in [5])

$$F_\psi = 2^a \widetilde{D}'/(\widetilde{D}, \widetilde{D}'), \tag{4.1}$$

where $a = 1$ if all of $\widetilde{D}$, $\widetilde{D}'$ and $\widetilde{D}''$ are even. Otherwise $a = 0$. The ray class character ψ' for $\mathbf{K}'$ is defined similarly, and its conductor is,

$$F_{\psi'} = 2^a \widetilde{D}/(\widetilde{D}', \widetilde{D}). \tag{4.2}$$

4.2 Galois groups and norm maps

We first identify the elements of the Galois group of the field $\mathbf{KK}'$ over $\mathbb{Q}$. In general, if $\mathbf{L}$ is a field extension of the field $\mathbf{L_0}$, the group of automorphisms of $\mathbf{L}$ which leaves every element of $\mathbf{L_0}$ fixed is called the *Galois group of* $\mathbf{L}$ *over* $\mathbf{L_0}$, and is denoted $\mathrm{Gal}(\mathbf{L}/\mathbf{L_0})$. Its order is at most the degree of the extension. In particular, the biquadratic extension $\mathbf{KK}' = \{a + b\sqrt{D} + c\sqrt{D'} + d\sqrt{D}\sqrt{D'} | a,\ b,\ c,\ d \in \mathbb{Q}\}$ has Galois group $\mathrm{Gal}(\mathbf{KK}'/\mathbb{Q}) = \{1,\ \delta,\ \delta',\ \gamma\}$, where

$$\delta : \begin{cases} \sqrt{D} & \mapsto & \sqrt{D} \\ \sqrt{D'} & \mapsto & -\sqrt{D'} \end{cases}, \quad \delta' : \begin{cases} \sqrt{D} & \mapsto & -\sqrt{D} \\ \sqrt{D'} & \mapsto & \sqrt{D'} \end{cases} \quad \text{and} \quad \gamma = \delta\delta'.$$

Thus γ acts as complex conjugation on both $\mathbf{K}$ and $\mathbf{K'}$ and thus gives the non-trivial element of the Galois group of either field over $\mathbb{Q}$. In what follows we shall use exponential notation for the action of Galois elements. Thus x^δ means x is acted on by δ and $x^{1-\gamma} = x(x^\gamma)^{-1}$.

Secondly, if $\mathbf{L} \supset \mathbf{L_0}$ is an extension of algebraic number fields, then there exist norm homomorphisms, both denoted by $\mathcal{N}_{\mathbf{L}/\mathbf{L}_0}$, from the group of units of $\mathbf{L}$ to the group of units of $\mathbf{L_0}$, and from the group of fractional ideals of $\mathbf{L}$ to the group of fractional ideals of $\mathbf{L_0}$. Thus,

$$\begin{array}{cccccccc} \mathcal{N}_{\mathbf{L}/\mathbf{L}_0}: & \mathbf{L}^\times & \to & \mathbf{L_0}^\times & :\lambda & \mapsto & \mathcal{N}_{\mathbf{L}/\mathbf{L}_0}(\lambda), \\ \mathcal{N}_{\mathbf{L}/\mathbf{L}_0}: & \mathcal{I}(\mathbf{L}) & \to & \mathcal{I}(\mathbf{L}_0) & :I & \mapsto & \mathcal{N}_{\mathbf{L}/\mathbf{L}_0}(I). \end{array}$$

These norms are related by the fact that they 'agree' on elements λ of $\mathbf{L}$. That is,

$$\mathcal{N}_{\mathbf{L}/\mathbf{L_0}}(\lambda \mathcal{O}_\mathbf{L}) = \mathcal{N}_{\mathbf{L}/\mathbf{L_0}}(\lambda)\mathcal{O}_{\mathbf{L_0}}.$$

Moreover, if $I \in \mathcal{I}(\mathbf{L})$, one has,

$$\mathcal{N}(I)\mathbb{Z} = \mathcal{N}_{\mathbf{L}/\mathbb{Q}}(I).$$

If the extension $\mathbf{L}$ over $\mathbf{L_0}$ is quadratic with $\mathrm{Gal}(\mathbf{L}/\mathbf{L_0}) = \{1,\ \sigma\}$, then, for $\lambda \in \mathbf{L}^\times$,

$$\mathcal{N}_{\mathbf{L}/\mathbf{L}_0}(\lambda) = \lambda^{1+\sigma}$$

and for $I \in \mathcal{I}(\mathbf{L})$, we have

$$\mathcal{N}_{\mathbf{L}/\mathbf{L}_0}(I)\mathcal{O}_\mathbf{L} = I^{1+\sigma}.$$

This equation determines the norm map on ideals since the map,

$$\mathcal{I}(\mathbf{L}_0) \to \mathcal{I}(\mathbf{L}) : J \mapsto J\mathcal{O}_\mathbf{L},$$

is injective. In particular, if $\mathbf{L} = \mathbf{KK'}$ and $\mathbf{L_0} = \mathbf{K}$, one has that, for $I \in \mathcal{I}(\mathbf{KK'})$ and $J \in \mathcal{I}(\mathbf{K})$,

$$\mathcal{N}_{\mathbf{KK'}/\mathbf{K}}(I)\mathcal{O}_{\mathbf{KK'}} = I^{1+\delta} \quad \text{and} \quad \mathcal{N}(J)\mathcal{O}_\mathbf{K} = J\overline{J}.$$

Finally, choose a fractional ideal F in $\mathcal{I}(\mathbf{K})$ which is contained in the conductor F_ψ (4.1), and which is self-conjugate, i.e.,

$$F \in \mathcal{I}(\mathbf{K}), \quad F \subset F_\psi \quad \text{and} \quad F^\gamma\ (= \overline{F})\ = F.$$

We define a subgroup A_F and a coset S_F in the ray class group $\mathcal{C}_F(\mathbf{K})$ by,

$$A_F \overset{\text{def}}{=} \ker(\psi_F)^{1-\gamma} = \{[I/\overline{I}^{-1}]_F \mid \psi_F([I]_F) = 1\}, \quad S_F \overset{\text{def}}{=} \{[I\overline{I}^{-1}]_F \mid \psi_F([I]_F) = -1\}. \tag{4.3}$$

For $F' \subset F_{\psi'}$ we define $A'_{F'}$ and $S'_{F'} \subset \mathcal{C}_{F'}(\mathbf{K'})$, similarly.

Note that if $\psi_F([I]_F) = -1$ then $S_F = (I/\overline{I})A_F$. Also if F_1 is a self-conjugate ideal contained in F, we find that,

$$A_F = \mathrm{red}_F^{F_1}(A_{F_1}) \text{ and } S_F = \mathrm{red}_F^{F_1}(S_{F_1}). \tag{4.4}$$

4.3 The three crucial theorems

To provide a link between the arithmetic of $\mathbf{K}$ and that of $\mathbf{K}'$ we need to give a correspondence between conductors in $\mathcal{O}_{\mathbf{K}}$ and in $\mathcal{O}_{\mathbf{K}'}$.

Definition 4.1 *We say that a pair (F, F') of self-conjugate ideals, $F \in \mathcal{I}(\mathbf{K})$ and $F' \in \mathcal{I}(\mathbf{K}')$ is* admissible *if $F \subset F_\psi$, $F' \subset F'_{\psi'}$ and $\mathcal{N}(F)\tilde{D} = \mathcal{N}(F')\tilde{D}'$.*

In this subsection we present our main theorem for producing identities, Theorem 4.3, which is a practical theorem relating ray class theta functions of different quadratic fields. Given an admissible pair (F, F'), the theorem describes coincidences between certain combinations of ray class theta functions of $\mathbf{K}$ with conductor F and similar combinations of ray class theta functions of $\mathbf{K}'$ with conductor F'.

The origin of these coincidences can be described in terms of L-functions of characters for the ray class groups $\mathcal{C}_F(\mathbf{K})$ and $\mathcal{C}_{F'}(\mathbf{K}')$. We first define such L-functions, then give a crucial relation between them in Theorem (4.2).

Given the definition (3.8) of theta functions for a formal complex linear combination of ray classes $\theta(W; d)$, one obtains the corresponding L-function $L(W)$ by using the modified Mellin transform, M_d, which sends q^t to $(td)^{-s}$,

$$M_d(\theta(W; d)) = L(W) = \sum_{x \in X} n_x L(x), \tag{4.5}$$

where $L(x) = \sum_{I \in x, I \subseteq \mathcal{O}_{\mathbf{K}}} \mathcal{N}(I)^{-s}$.

Now, if χ is a ray class character of $\mathbf{K}$ with conductor F_χ dividing F, we may define the *L-function of the ray class character χ with conductor F* as follows,

$$L_F(\chi) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{C}_F(\mathbf{K})} \chi(x) L(x) = \sum_{I \in \mathcal{I}_F(\mathbf{K}), I \subseteq \mathcal{O}_{\mathbf{K}}} \chi([I]_F) \mathcal{N}(I)^{-s}.$$

We put $L(\chi) = L_{F_\chi}(\chi)$. We now have,

Theorem 4.2 *Let (F, F') be an admissible pair and let χ and χ' be characters of $\mathcal{C}_F(\mathbf{K})$ and $\mathcal{C}_{F'}(\mathbf{K}')$ such that*

(i) χ is 1 on A_F and -1 on S_F.

(ii) For all $I \in \mathcal{I}_{FF'\mathcal{O}_{\mathbf{KK}'}}(\mathbf{KK}')$, $\chi([\mathcal{N}_{\mathbf{KK}'/\mathbf{K}}(I)]_F) = \chi'([\mathcal{N}_{\mathbf{KK}'/\mathbf{K}'}(I)]_{F'})$.

Then $L_F(\chi) = L_{F'}(\chi')$.

The proof relies on the theory of Artin L-functions. In fact, for a suitable extension $\mathbf{N}$ of $\mathbf{KK}'$, the characters χ and χ' define, under the Artin correspondence of class field theory, characters χ_{gal} of $\Delta = \text{Gal}(\mathbf{N}/\mathbf{K})$ and χ'_{gal} of $\Delta' = \text{Gal}(\mathbf{N}/\mathbf{K}')$. Now the Artin L-functions of χ_{gal} and χ'_{gal} are, by definition, the L-functions of χ and χ',

$$L(\chi_{\text{gal}}) = L(\chi), \qquad L(\chi'_{\text{gal}}) = L(\chi').$$

One proves [5] that, under the conditions of the theorem, $\chi_{\rm gal}$ and $\chi'_{\rm gal}$ induce the same character of $\Gamma = \mathrm{Gal}(\mathbf{N}/\mathbb{Q})$,

$$\chi_{\rm gal}{}_{\uparrow_\Delta^\Gamma} = \chi'{}_{\rm gal}{}_{\uparrow_{\Delta'}^\Gamma}.$$

Since their Artin L-functions coincide with that of the induced character, one must have,

$$L(\chi) = L(\chi_{\rm gal}) = L(\chi'_{\rm gal}) = L(\chi'). \tag{4.6}$$

The result follows.

We are now ready to state the main theorem of this paper. It is obtained by summing (multiplied by suitable roots of unity) all the instances of (4.6) for a given admissible pair of conductors (a sort of finite Fourier inversion) and applying the inverse of the Mellin transform given in (4.5).

Theorem 4.3 *Let (F, F') be admissible and let I be an integral ideal of $\mathcal{O}_{\mathbf{KK}'}$ having no common factor with $FF'\mathcal{O}_{\mathbf{KK}'}$. Put $J = \mathcal{N}_{\mathbf{KK}'/\mathbf{K}}(I)$ and $J' = \mathcal{N}_{\mathbf{KK}'/\mathbf{K}'}(I)$. Then*

$$\theta(A_F[J]_F; d) - \theta(S_F[J]_F; d) = \theta(A'_{F'}[J']_{F'}; d) - \theta(S'_{F'}[J']_{F'}; d), \tag{4.7}$$

for $d \in \mathbb{R}$.

Unfortunately, as will be exemplified in the next section when we discuss the first set of Virasoro character identities (2.4), coset theta functions may give rise, by application of (3.9), to theta functions of ray classes with respect to non-self-conjugate conductors. The following theorem describes situations where such theta functions may be replaced by theta functions of ray classes with self-conjugate conductors. It relies on the cancellation available from the relationship (for self-conjugate F)

$$L([\,\overline{I}\,]_F) = L([I]_F).$$

Theorem 4.4 *Let F, P and J be (integral) ideals of $\mathcal{O}_{\mathbf{K}}$. Suppose that F is self conjugate, that P is maximal and prime to F and that J is prime to PF.*

Let B be a subgroup of $\mathcal{C}_F(\mathbf{K})$ containing both $[P/\overline{P}]_F^2$ and $[J/\overline{J}]_F$. Put T for the coset $B[P/\overline{P}]_F$ and put $\tilde{B}$ and $\tilde{T}$ for the inverse images of B and T in $\mathcal{C}_{FP}(\mathbf{K})$.

Then

$$\theta(\tilde{B}[J]_{FP}; d) - \theta(\tilde{T}[J]_{FP}; d) = \theta(B[J]_F; d) - \theta(T[J]_F; d). \tag{4.8}$$

We shall apply this result in cases where $B = A_F$ and $T = S_F$. We note that in this case, the conditions of the second paragraph of the above theorem hold, provided $f_\psi \mid F$, $\psi(J) = 1$ and $\psi(P) = -1$.

The proofs of the above theorems (or rather the corresponding theorems for L-functions) may be found in [5]. We now use them to prove the Virasoro identities (2.4, 2.5), but also to uncover a whole family of identities between Virasoro characters at higher levels, of which the identities (2.4) are the simplest example.

5 Virasoro identities

In the next three subsections, we use the algebraic tools developed in this paper to prove the identities (2.4). We first show in subsection 5.1. how to relate ray class theta functions associated with the two relevant quadratic fields $\mathbf{K} = \mathbb{Q}[\sqrt{-2}]$ and $\mathbf{K}' = \mathbb{Q}[\sqrt{-1}]$ using Theorem 4.3. We then rewrite the identities as in (2.6). To make contact with Theorem 4.3, we express the left hand side (resp. the right hand side) of (2.6) in terms of differences of ray class theta functions for $\mathcal{C}_F(\mathbf{K})$ (resp. $\mathcal{C}_{F'}(\mathbf{K}')$) for self-conjugate conductors F (resp. F'). This is carried out in detail in subsections 5.2. and 5.3.

Remarkably, the three distinct differences of ray class theta functions appearing in the RHS of (2.6) are obtained each time one considers quadratic expressions in Virasoro unitary minimal characters at level $m = 4a^2$ for a odd and $1+4a^2 = a'^2 p$ with p prime, in a way described in Theorem (5.1), subsection 5.3. This does then provide an infinite class of new identities between the Virasoro characters at level $m = 3$ and Virasoro characters at level $m = 36, 100, 196, \ldots$ We also remark (5.23) how the LHS of these identities may be rewritten as sums of Virasoro characters at any of an infinite series of higher levels $m = 675, 131043, \ldots$

The subsection 5.4. gives a somewhat terser account of how to prove the second set of identities (2.5). There, the two relevant quadratic fields are $\mathbf{K} = \mathbb{Q}[\sqrt{-30}]$ and $\mathbf{K}' = \mathbb{Q}[\sqrt{-10}]$.

We recall (see Section 3) that in what follows, P_p (resp. P'_p) stands for a maximal ideal of $\mathcal{O}_\mathbf{K}$ (resp. of $\mathcal{O}_{\mathbf{K}'}$) of norm p when p is prime.

5.1 Relations between ray class theta functions

As already remarked at the end of Section 3, the expression on the left hand side of the identities (2.6) is associated with the quadratic field $\mathbb{Q}[\sqrt{D}]$ with $D = -2$, while the right hand side is associated with the quadratic field $\mathbb{Q}[\sqrt{D'}]$ with $D' = -1$. So, in the notations adopted in this paper, one has $\mathbf{K} = \mathbb{Q}[\rho]$ with $\rho = \sqrt{-2}$, $\mathcal{O}_\mathbf{K} = \langle 1, \rho\rangle_{gp}$, and the units of $\mathcal{O}_\mathbf{K}$ are $\mathcal{O}_\mathbf{K}{}^\times = \{\pm 1\}$. Also, $\mathbf{K}' = \mathbb{Q}[i]$, $\mathcal{O}_{\mathbf{K}'} = \langle 1, i\rangle_{gp}$ and $\mathcal{O}_{\mathbf{K}'}{}^\times = \{\pm 1, \pm i\}$.

Consider the two ray class characters ψ and ψ' defined in subsection 4.1. Their conductors are given by the expressions (4.1,4.2) with $\tilde{D} = -8$, $\tilde{D}' = -4$ (and $\widetilde{D''} = 8$, so $a = 1$). Thus,

$$F_\psi = \left(\tfrac{2[8,4]}{8}\right)_{\mathcal{O}_\mathbf{K}} = (2)_{\mathcal{O}_\mathbf{K}}$$
$$F_{\psi'} = \left(\tfrac{2[4,8]}{4}\right)_{\mathcal{O}_{\mathbf{K}'}} = (4)_{\mathcal{O}_{\mathbf{K}'}}.$$

(Note that, especially in subscripts, we shall write principal ideals (α) instead of $\alpha\mathcal{O}_\mathbf{K}$ or $\alpha\mathcal{O}_{\mathbf{K}'}$).

Both $\mathcal{O}_\mathbf{K}$ and $\mathcal{O}_{\mathbf{K}'}$ are principal ideal domains, i.e. all their ideals are principal, so that, using (3.11), we can easily identify the ray class groups of $\mathbf{K}$ and $\mathbf{K}'$ associated

with the above conductors,

$$\mathcal{C}_{F_\psi}(\mathbf{K}) = \mathcal{C}_{(2)}(\mathbf{K}) = \mathcal{CP}_{(2)}(\mathbf{K}) \simeq (\mathcal{O}_{\mathbf{K}}/(2))^\times = \langle [1+\rho]_{(2)} \rangle_{gp}$$
$$\mathcal{C}_{F_{\psi'}}(\mathbf{K}') = \mathcal{C}_{(4)}(\mathbf{K}') = \mathcal{CP}_{(4)}(\mathbf{K}') \simeq (\mathcal{O}_{\mathbf{K}'}/(4))^\times/\{\pm 1, \pm i\} = \langle [1+2i]_{(4)} \rangle_{gp}.$$

Both groups are of order 2, and so ψ and ψ' are the only non trivial (1-dimensional) characters of $\mathcal{C}_{F_\psi}(\mathbf{K})$ and $\mathcal{C}_{F_{\psi'}}(\mathbf{K}')$. We thus have,

$$\begin{aligned} \psi([\alpha]_{(2)}) &= 1 \quad \text{if } \alpha \equiv 1 \mod (2) \\ &= -1 \quad \text{otherwise} \end{aligned} \tag{5.1}$$

$$\begin{aligned} \psi'([\alpha]_{(4)}) &= 1 \quad \text{if } \alpha \equiv \pm 1 \text{ or } \pm i \mod (4) \\ &= -1 \quad \text{otherwise.} \end{aligned} \tag{5.2}$$

The next step is to calculate the A's and S's of (4.3) for the admissible pairs of ideals (F, F') (see Definition 4.1) that we shall be using. Put $P_2 = (\rho)_{\mathcal{O}_\mathbf{K}}$ as before and $P_2' = (1+i)_{\mathcal{O}_{\mathbf{K}'}}$, the prime ideal of norm 2 in $\mathcal{O}_{\mathbf{K}'}$ (noting that $P_2'^2 = 2\mathcal{O}_{\mathbf{K}'}$). Then $(4P_2, (8)_{\mathcal{O}_{\mathbf{K}'}})$ and $((4)_{\mathcal{O}_\mathbf{K}}, 4P_2')$ are admissible pairs for $\mathbf{K}$ and $\mathbf{K}'$. We concentrate on the first pair as the data for the second pair will come easily by (4.4). By (5.1),

$$\ker\ \psi_{4P_2} = \{[1+2\alpha]_{4P_2} \mid \alpha \in \mathcal{O}_\mathbf{K}\} = \{[1+2a+2b\rho]_{4P_2} \mid a, b \in \mathbb{Z}\}.$$

Thus, in this case, the A-group is trivial,

$$A_{4P_2} = (\ker \psi_{4P_2})^{1-\gamma} = \{[1]_{4P_2}\}, \tag{5.3}$$

since $\overline{1+2a+2b\rho} = 1+2a-2b\rho \equiv 1+2a+2b\rho \mod 4P_2\,(=(4\rho))$ (and γ acts as conjugation). In order to obtain the coset S_{4P_2} (Definition (4.3)), we must choose an ideal I in $\mathcal{I}_{4P_2}(\mathbf{K})$ such that $\psi\,[I]_{4P_2} = -1$. We take $I = (1+\rho)_{\mathcal{O}_\mathbf{K}}$. Then we find that S_{4P_2} consists of the class

$$[I/\bar{I}]_{4P_2} = [(1+\rho)(1-\rho)^{-1}]_{4P_2} = [3(-1+2\rho)/9]_{4P_2} = [-3+2\rho]_{4P_2} = [5 \pm 2\rho]_{4P_2}. \tag{5.4}$$

We also need $A'_{(8)}$ and its coset $S'_{(8)}$ for $\mathbf{K}'$. Note that now,

$$\ker\ \psi'_{(8)} = \{[1+4\alpha]_{(8)} \mid \alpha \in \mathcal{O}_{\mathbf{K}'}\} = \{[1+4a+4bi]_{(8)} \mid a, b \in \mathbb{Z}\}, \tag{5.5}$$

and so

$$A'_{(8)} = \ker\ \psi'_{(8)})^{1-\gamma} = \{[1]_{(8)}\}, \tag{5.6}$$

since, modulo (8), $1+4a+4bi \equiv 1+4a-4bi = \overline{1+4a+4bi}$. Moreover, with $I = (1+2i)_{\mathcal{O}_{\mathbf{K}'}}$, $\psi'[I]_{(8)} = -1$, and so we find that $S'_{(8)}$ consists of the class

$$[I/\bar{I}]_{(8)} = [(1+2i)(1-2i)^{-1}]_{(8)} = [5(1+2i)^2/25]_{(8)} = [-1+4i]_{(8)} = [1+4i]_{(8)}. \tag{5.7}$$

For the second pair of conductors we find that the ideal $(4)_{\mathcal{O}_{\mathbf{K}}}$ divides the ideal $4P_2$, and $4P_2'$ divides $(8)_{\mathcal{O}_{\mathbf{K}'}}$. So we can obtain the new A's and S's by reduction using (4.4)

$$
\begin{aligned}
A_{(4)} = \{[1]_{(4)}\} \quad \text{and} \quad S_{(4)} = [1+2\rho]_{(4)}, \\
A'_{4P_2'} = (\ker\ \psi'_{4P_2'})^{1-\gamma} = \{[1]_{4P_2'}\} \quad \text{and} \quad S'_{4P_2'} = [5]_{4P_2'} = [3]_{4P_2'}.
\end{aligned} \tag{5.8}
$$

Now take I, F and F' in Theorem 4.3 to be successively

$$
\begin{array}{cccc}
\mathcal{O}_{\mathbf{KK}'}, & 4P_2 & \text{and} & 8\mathcal{O}_{\mathbf{K}'}; \\
(1-\rho)_{\mathcal{O}_{\mathbf{KK}'}}, & 4P_2 & \text{and} & 8\mathcal{O}_{\mathbf{K}'}; \quad \text{and} \\
\mathcal{O}_{\mathbf{KK}'}, & 4\mathcal{O}_{\mathbf{K}}, & \text{and} & 4P_2'.
\end{array}
$$

This gives the following relations between ray class theta functions of the fields $\mathbb{Q}[\rho]$ and $\mathbb{Q}[i]$,

$$
\begin{aligned}
\theta([1]_{4P_2};16) - \theta(S_{4P_2};16) &= \theta([1]_{(8)};16) - \theta(S'_{(8)};16) \\
\theta([1+2\rho]_{4P_2};16) - \theta([1+2\rho]_{4P_2}S_{4P_2};16) &= \theta([3]_{(8)};16) - \theta([3]_{(8)}S'_{(8)};16) \\
\theta([1]_{(4)};8) - \theta(S_{(4)};8) &= \theta([1]_{4P_2'};8) - \theta(S'_{4P_2'};8).
\end{aligned} \tag{5.9}
$$

Here, for the second line we have calculated the norms of $(1-\rho)$ from $\mathbf{KK}'$ to $\mathbf{K}$ and from $\mathbf{KK}'$ to $\mathbf{K}'$ as follows

$$
\begin{aligned}
\mathcal{N}_{\mathbf{KK}'/\mathbf{K}}(1-\rho) = (1-\rho)^2 = -(1+2\rho), \\
\mathcal{N}_{\mathbf{KK}'/\mathbf{K}'}(1-\rho) = (1-\rho)(1+\rho) = 3.
\end{aligned}
$$

5.2 Reduction of V products to ray class theta functions

We have shown at the end of section 3 how to rewrite a product of two generalised theta functions as a ray class theta function, namely,

$$
\theta_{(1+6a),6}\theta_{s(1+6b),12} = \theta\left(\left[\frac{\alpha_s(a,b)}{\rho(1-\rho)}\right]_{P_3F};16\right), \tag{5.10}
$$

with $s \equiv 1 \bmod 6$ and $F = 4P_2$. From (2.7) and the above relation, we obtain the V product $V(1,2)V(s,3)$ as a linear combination of ray class theta functions,

$$
V(1,2)V(s,3) = (\theta_{1,6} - \theta_{7,6})(\theta_{s,12} - \theta_{7s,12}) = \theta(W;16), \tag{5.11}
$$

where

$$
W = [\beta_s]_{P_3F}(q(0,0) + q(1,1) - q(1,0) - q(0,1)) \in \mathcal{C}_{P_3F}(\mathbf{K}), \tag{5.12}
$$

with $\beta_s = \alpha_s(0,0)/((1-\rho)\rho)$ and $q(a,b) = [\alpha_s(a,b)/\alpha_s(0,0)]_{P_3F}$.

We express the elements $[\delta]_{P_3F}$ of $\mathcal{C}_{P_3F}(\mathbf{K}) = \mathcal{CP}_{P_3F}(\mathbf{K})$ in the form $[\alpha,\beta]_{P_3F}$ as explained after (3.13), following the decomposition

$$
(\mathcal{O}_{\mathbf{K}}/P_3F)^\times \cong (\mathcal{O}_{\mathbf{K}}/P_3)^\times \times (\mathcal{O}_{\mathbf{K}}/F)^\times.
$$

In order to apply Theorem 4.3 we shall combine the ray classes of (5.11) to make classes with respect to the self-conjugate conductor F using Theorem 4.4. We take P, B and T of that theorem to be, respectively, P_3, A_F ($= \{[1]_F\}$ by 5.3) and S_F ($= \{[5-2\rho]_F\}$ by 5.4). The choice of $P = (1+\rho)$ and T are consistent since from (5.1), $\psi([1+\rho]) = -1$. We first identify $\tilde{B}$ and $\tilde{T}$ and then compare them to the classes $q(a,b)$.

We know that $\mathcal{N}(P_3) = 3$. So $(\mathcal{O}_{\mathbf{K}}/P_3)^\times = \{\pm 1 + P_3\}$ and by, (3.15), $\tilde{B}$ and $\tilde{T}$ of Theorem 4.4 are $\{[\pm 1, 1]_{P_3F}\}$ and $\tilde{B}[5-2\rho]_{P_3F}$, respectively.

Now $\alpha_s(1,1) = 7\alpha_s(0,0)$. So

$$q(1,1) = [7]_{P_3F} = [1,-1]_{P_3F} = [-1,1]_{P_3F}, \tag{5.13}$$

by (3.14). Also, $7\alpha_s(1,0) = 98 + 7s\rho \equiv \alpha_s(0,1) \bmod 24$. So

$$\alpha_s(0,1)/(\rho(1-\rho)) \equiv 7\alpha_s(1,0)/(\rho(1-\rho)) \bmod P_3F$$

and hence

$$q(0,1) = [7]_{P_3F}q(1,0) = [1,-1]_{P_3F}q(1,0).$$

Again, $\alpha_s(0,0)(5-2\rho) \equiv \cdots \equiv \alpha_s(1,0) \bmod 24$. So, in the same way,

$$q(1,0) = [5-2\rho]_{P_3F}q(0,0).$$

Thus

$$\{q(0,0), q(1,1)\} = \tilde{B} \quad \text{and} \quad \{q(1,0), q(0,1)\} = \tilde{T}$$

Putting this information into (5.12) we get, from 5.11,

$$V(1,2)V(|s|,3) = \theta([\beta_s]_{P_3F}\tilde{B};16) - \theta([\beta_s]_{P_3F}\tilde{T};16) = \theta([\beta_s]_F;16) - \theta([\beta_s]_F S_F;16) \tag{5.14}$$

by Theorem 4.4. Moreover, $\beta_s = 1$ if $s = 1$ and $\beta_s = 1 + 2\rho$ if $s = 5$.

If $s = -2$ one can divide the coset in (3.10) by $-\rho$ to get,

$$\theta_{(1+6a),6}\theta_{2(1+6b),12} = \theta(2(1+6b) - (1+6a)\rho + (J/P_2);8).$$

Here we have the same situation as before with $s_{\text{new}} = 1$ except that the rôoles of a and b are reversed, the ray class conductor is $4P_3$ and the scale factor 8 instead of 16. The same analysis goes through with $F = 4\mathcal{O}_{\mathbf{K}}$ and we obtain

$$V(1,2)V(2,3) = \theta([1]_{(4)};8) - \theta(S_{(4)};8). \tag{5.15}$$

Thus we have shown that the left hand sides of the identities (2.6) may be rewritten as the left hand sides of the identities (5.9). To complete our proof of (2.6) we shall, in a similar manner, show that the right hand sides of (2.6), which may be described by the V products

$$V(r,4)V(rf,4) + V(7r,4)V(7rf,4) \tag{5.16}$$

for $(r,f) = (1,2)$, $(3,2)$ and (1,3), are equal to the right hand sides of the identities (5.9).

5.3 An infinite family of identities

In fact the expressions (5.16) are only the first set of an infinite family of quadratic expressions in the V functions which reduce to the right hand sides of (5.9). We now prove this reduction for the whole family and thus obtain (by 2.9) an infinite family of identities between Virasoro characters at level $m = 3$ and products of those at levels $m = 4a^2$ where a is odd and $1 + 4a^2 = pa'^2$ with p prime.

Theorem 5.1 *Let a, a' and p be integers such that $a \equiv 1 \bmod 4$, p is prime and $4a^2 + 1 = pa'^2$. Put $m = 4a^2$ and $c = aa'$. Then, for r odd and prime to p and $\epsilon = 0$ or 1,*

$$\sum_{u=1}^{\frac{p-1}{2}} \sum_{v=0}^{c-1} \sum_{w=0}^{c-1} V(c\hat{u}(r+8vp), m) V(c\hat{u}((2a-\epsilon p)r+8wp), m) = \theta([r]_{F'} - [r\delta]_{F'}, 2^{4-\epsilon}), \quad (5.17)$$

where $\hat{u} = (u + 5p(1-u))$ and the ray classes on the right are defined in $\mathbf{K}' = \mathbb{Q}[i]$ with $F' = P_2'^{6-\epsilon}$, $P_2' = (1+i)\mathcal{O}_{\mathbf{K}'}$ (as in subsection 5.1) and $\delta = 1 + 4i$ so that $\{[\delta]_{F'}\} = S'_{F'}$ (see 5.7 and 5.8).

Actually, the theorem gives just three different relations. These correspond to the choices $(r, \epsilon) = (1, 0)$, $(3, 0)$ and $(1, 1)$ and have as right hand sides the three right hand sides of (5.9). Moreover if we take $a = 1$ and $p = 5$, so $m = 4$, $a' = c = 1$ and $\hat{2} = -23$ and put $f = |2a - \epsilon p| = 2$ or 3, then the LHS in Theorem 5.1 becomes,

$$V(r, 4)V(rf, 4) + V(23r, 4)V(23rf, 4).$$

Now we may make this expression more economical by replacing 23 by 7 (since $23 \times 9 = 207 \equiv 7$ modulo $2m(m + 1) = 40$, $9 = 2m + 1$ and $V(r(2m + 1), m) = -V(r, m)$). So we have,

$$V(r, 4)V(rf, 4) + V(7r, 4)V(7rf, 4) = \theta([r]_{\tilde{F}'}; 2^{4-\epsilon}) - \theta([r]_{\tilde{F}'} S'_{\tilde{F}'}; 2^{4-\epsilon}). \qquad (5.18)$$

Taking $(r, f) = (1, 2), (3, 2)$ and $(1, 3)$, and using (5.14), (5.15) and (5.9) we get the identities (2.6).

Before we can prove Theorem 5.1, we need some preparation.

Suppose that $\Lambda \supset \Lambda'$ are lattices in the inner product space V. Then there is a subset $T \subset \Lambda$ (a transversal of Λ over Λ') of the same size as the quotient group Λ/Λ' such that $(w + \Lambda')$ and $(w' + \Lambda')$ are disjoint for distinct w and w' in T. Then, for any $v \in V$,

$$v + \Lambda = \bigcup_{w \in T} (v + w) + \Lambda'$$

and so,

$$\theta(v + \Lambda; d) = \sum_{w \in T} \theta((v + w) + \Lambda'; d). \qquad (5.19)$$

Lemma 5.2 *Suppose that $k = c^2k'$ with c and $k' \in \mathbb{N}$. Choose $b \in \mathbb{Z}$ to have no common factor with c. Then*

$$\sum_{j=0}^{c-1} \theta_{cb(r+2jk'),k} = \theta_{br,k'}. \tag{5.20}$$

Proof: Now

$$\theta_{cb(r+2jk'),k} = \theta\left(\frac{cb(r+2jk')}{2k} + \mathbb{Z};\ \frac{1}{k}\right) = \theta\left(\left(\frac{br}{2k'} + bj\right) + c\mathbb{Z};\ \frac{1}{k'}\right).$$

Whereas

$$\theta_{br,k'} = \theta\left(\frac{br}{2k'} + \mathbb{Z}; \frac{1}{k'}\right).$$

But since b is invertible mod c, the set $\{0,\ b, \ldots, (c-1)b\}$ is a transversal for $\mathbb{Z}$ over $c\mathbb{Z}$. So the identity follows by (5.19). As an immediate consequence we have,

Corollary 5.3 *If $m(m+1) = k$ above then*

$$\sum_{j=0}^{c-1} V(cb(r+2jk'), m) = \theta_{br,k'} - \theta_{br(2m+1),k'}. \tag{5.21}$$

We remark, by the way, that taking $m = 242$, $c = 99$, $k' = 6$ and $b = r = 1$ gives a RHS in (5.21) of $\theta_{1,6} - \theta_{5,6} = \eta$ and we obtain,

$$\sum_{j=0}^{98} \chi^{Vir(242)}_{99(1+12j),99(1+12j)} = 1. \tag{5.22}$$

This is the first in an infinite series of such identities (though, presumably not the first with RHS 1). The next, however has $m = 23762$ and $c = 109 \times 89 = 9701$.

Again, we may solve the Pellian equation

$$(2m+1)^2 - 48c^2 = 1 \quad \text{(i.e.} \quad m(m+1) = 12c^2)$$

and choose one of the (infinitely many) solutions such that $2m + 1 \equiv 7$ modulo 24. (e.g. $m = 675$, $c = 175$; $m = 131043$, $c = 37829$.) In Lemma 5.2, we now have $k' = 12$ and $b = 1$ gives a RHS in (5.21) of $\theta_{r,12} - \theta_{7r,12} = V(r, 3)$ and we obtain, for instance,

$$\sum_{j=0}^{174} \chi^{Vir(675)}_{175(1+24j),175(1+24j)} = \chi^{Vir(3)}_{r,r}. \tag{5.23}$$

This is the first in an infinite series of such identities which (taking $r = 1$, -2 and -5) rewrite the left hand sides of (2.4).

Proof of Theorem 5.1 : Note the congruences $2a \equiv 2 \bmod 8$ and $p \equiv 5 \bmod 8$ (so also $5p \equiv 1 \bmod 8$). In particular, $\hat{u} \equiv u \bmod p$ and $\hat{u} \equiv 1 \bmod 8$.

(i) We first rewrite the left hand side of (5.17) as a sum of differences of coset theta functions.

Now $2m+1 = 2p(a')^2 - 1 \equiv 2p-1$ modulo $8p$. So by (5.21), the uth term in the outer summation on the LHS of (5.17) is

$$\mathrm{LHS}(u) = (\theta_{r\hat{u},4p} - \theta_{r\hat{u}(2p-1),4p})(\theta_{(2a-\epsilon p)r\hat{u},4p} - \theta_{(2a-\epsilon p)r\hat{u}(2p-1),4p}).$$

Applying (2.13), (with $k = l = h = 4p$, $\lambda = \mu = k_0 = \ell_0 = 1$ and $D = -1$)

$$\begin{aligned}\mathrm{LHS}(u) \;&=\; \theta(r\hat{u}\alpha_1 + 8p\mathcal{O}_{\mathbf{K}'}; 16p) - \theta(r\hat{u}\alpha_2 + 8p\mathcal{O}_{\mathbf{K}'}; 16p)\\ &\qquad -\theta(r\hat{u}\alpha_3 + 8p\mathcal{O}_{\mathbf{K}'}; 16p) + \theta(r\hat{u}\alpha_4 + 8p\mathcal{O}_{\mathbf{K}'}; 16p), \end{aligned} \tag{5.24}$$

where,

$$\begin{aligned}&\alpha_1 = 1 + (2a - p\epsilon)i;\\ &\alpha_2 = 1 - (2a - p\epsilon)(2p-1)i);\\ &\alpha_3 = (2p-1) - (2a - p\epsilon)i \equiv (2p-1)\alpha_2 \bmod 8p;\\ &\alpha_4 = (2p-1)(1 + (2a - p\epsilon)i) = (2p-1)\alpha_1.\end{aligned}$$

(Note that $(2p-1)^2 \equiv 1 \bmod 8p$.) Now $\hat{u}$ is $u \bmod p$ and $1 \bmod 8$. So

$$\hat{u}(2p-1) \equiv -u \equiv p - u \equiv \widehat{p-u} \bmod p$$

and

$$\hat{u}(2p-1) \equiv 1(10-1) \equiv 1 \equiv \widehat{p-u} \bmod 8$$

and therefore

$$\hat{u}(2p-1) \equiv \widehat{p-u} \bmod 8p.$$

Thus

$$r\hat{u}\alpha_4 \equiv r(\widehat{p-u})\alpha_1 \quad \text{and} \quad r\hat{u}\alpha_3 \equiv r(\widehat{p-u})\alpha_2.$$

Hence the LHS of (5.17) may be rearranged as

$$\sum_{u=1}^{\frac{p-1}{2}} \mathrm{LHS}(u) = \sum_{u=1}^{p-1} \left(\theta(r\hat{u}\alpha_1 + 8p\mathcal{O}_{\mathbf{K}'}; 16p) - \theta(r\hat{u}\alpha_2 + 8p\mathcal{O}_{\mathbf{K}'}; 16p)\right), \tag{5.25}$$

since doubling the range of the sum exactly compensates for the elimination of the last two terms of (5.24). We now prepare to express these coset theta functions as ray class theta functions using (3.9) and to reduce the natural conductors using Theorem 4.4.

We note first that $p\mathcal{O}_{\mathbf{K}'} = P\overline{P}$, where $P = (1-2ai)_{\mathcal{O}_{\mathbf{K}'}}$. So (cf. 3.5), P and $\overline{P}$ are distinct prime ideals of norm p and by (3.16),

$$(\mathcal{O}_{\mathbf{K}'}/P)^\times = \{u + P \mid u \in \mathbb{Z},\ 1 \le u \le p-1\}. \tag{5.26}$$

We shall apply Theorem 4.4 with $\mathbf{K}$, P, F, B and T of that theorem being respectively $\mathbf{K}'$, P, $F' = (P_2')^{6-\epsilon}$, $A'_{F'}$ ($= \{[1]_{F'}\}$ by 5.6) and $S'_{F'}$ ($= \{[\delta]_{F'}\}$ by 5.7). The

choice of P and T are consistent since $1-2ai \equiv 1-2i$ mod 4 and so, from (5.2), $\psi([P]) = -1$.

We express the elements of $\mathcal{C}_{PF'}(\mathbf{K}')$ in the form $[\alpha,\beta]_{PF'}$ as explained after (3.13), following the decomposition

$$(\mathcal{O}_{\mathbf{K}'}/PF')^{\times} \cong (\mathcal{O}_{\mathbf{K}'}/P)^{\times} \times (\mathcal{O}_{\mathbf{K}'}/F')^{\times},$$

Then, by (3.15) and (5.26), $\tilde{B}$ of Theorem 4.4 is

$$\tilde{B} \stackrel{\text{def}}{=} \left(\text{red}_{F'}^{PF'}\right)^{-1}(B) = \{[u,1]_{PF'} \mid 1 \le u \le p-1\} = \{[\hat{u}]_{PF'} \mid 1 \le u \le p-1\}$$

and $\tilde{T} = \tilde{B}[\delta']_{PF'}$, provided $[\delta']_{F'} = [\delta]_{F'}$.

(ii) We examine first the case when $\epsilon = 0$. Then $\alpha_1\mathcal{O}_{\mathbf{K}'} = \overline{P}$ and so, for $\beta \in \mathcal{O}_{\mathbf{K}'}$ prime to $2P$, the highest common factor H of $(\beta\alpha_1)_{\mathcal{O}_{\mathbf{K}'}}$ and $(8p)_{\mathcal{O}_{\mathbf{K}'}} = 8P\overline{P}$ is $\overline{P}$. Therefore, using the relation (3.9) between coset and ray class theta functions for the ray class group $\mathcal{C}_{PF'}(\mathbf{K}')$, we get,

$$\theta(\beta\alpha_2 + 8p\mathcal{O}_{\mathbf{K}'}; 16p) = \theta([\beta]_{PF'}; 16). \tag{5.27}$$

Now $\tilde{T} = \tilde{B}[\alpha_2/\alpha_1]_{PF'}$, since $\alpha_2/\alpha_1 = 1-4(1-2ai)i$ is prime to P and congruent to δ modulo 8. Thus, applying (5.27) to (5.25) (with $\beta = r\hat{u}$ or $r\hat{u}\alpha_2/\alpha_1$) we find that the LHS of (5.17) is

$$\begin{aligned}\sum_{u=1}^{p-1}\left(\theta([r\hat{u}]_{PF'};16) - \theta([r\hat{u}\alpha_2/\alpha_1]_{PF'};16)\right) &= \theta(\tilde{B}[r]_{PF'},16) - \theta(\tilde{T}[r]_{PF'},16) \\ &= \theta([r]_F - [r\delta]_F; 16),\end{aligned} \tag{5.28}$$

by Theorem 4.4, as required.

(iii) Now consider the case when $\epsilon = 1$.

For $\beta \in \mathcal{O}_{\mathbf{K}'}$ prime to $2P$, the highest common factor, H, of $(\beta(1+2ai)(1+i))_{\mathcal{O}_{\mathbf{K}'}} = \beta\overline{P}P_2$ and $(8p)_{\mathcal{O}_{\mathbf{K}'}} = P_2'^6 P\overline{P}$ is $\overline{P}P_2'$. Therefore, using (3.9) again, we get

$$\theta(\beta(1+2ai)(1+i) + 8p\mathcal{O}_{\mathbf{K}'}; 16p) = \theta([\beta]_{PF'}; 8), \tag{5.29}$$

Put $\beta_j = \alpha_j/((1+2ai)(1+i))$, for $j = 1, 2$. Then, since $2a \equiv 2$ mod 8,

$$(1+i)\beta_1 = \frac{1+2ai-pi}{(1+2ai)} = 1-(1-2ai)i \equiv -1-i \text{ mod } 8.$$

So $\beta_1 \equiv -1$ modulo $(8/(1+i))_{\mathcal{O}_{\mathbf{K}'}} = F'$ and so $[\beta_1]_{F'} = [1]_{F'}$ and (from the third expression) β_1 is prime to P. Thus $\tilde{B}[\beta_1]_{PF'} = \tilde{B}$.

Again

$$(1+i)\beta_2 = \frac{1+2ai-ip(1-2p+4a)}{(1+2ai)} = 1-i(1-2ai)(1-2p+4a).$$

So β_2 is prime to P and $(1+i)\beta_2 \equiv 1-(i+2)(1-2+4) \equiv -2-3(1+i)$ modulo 8. So

$$\beta_2 \equiv -(1-i)-3 = -(4+i) \equiv -i\delta \bmod 4P_2'.$$

Hence $\tilde{B}[\beta_2]_{PF'} = \tilde{T}$. Thus, applying (5.29) to (5.25) (with $\beta = r\hat{u}\beta_1$ or $r\hat{u}\beta_2$) we find that the LHS of (5.17) is

$$\sum_{u=1}^{p-1}(\theta([r\hat{u}\beta_1]_{PF'};8)-\theta([r\hat{u}\beta_2]_{PF'};8)) = \theta(\tilde{B}[r]_{PF'},8)-\theta(\tilde{T}[r]_{PF'},8) = \theta([r]_F-[r\delta]_F;8),$$

by Theorem 4.4, as required.

5.4 The second set of identities

The proof of the second set of identities (2.5) relies on Theorem 4.3, as did the proof of the identities (2.4) in subsection 5.1. However now, the relevant quadratic fields are $\mathbf{K} = \mathbb{Q}[\sqrt{-30}]$ and $\mathbf{K}' = \mathbb{Q}[\sqrt{-10}]$. By (3.1), their rings of algebraic integers are $\mathcal{O}_{\mathbf{K}} = \mathbb{Z}[\sqrt{-30}]$ and $\mathcal{O}_{\mathbf{K}'} = \mathbb{Z}[\sqrt{-10}]$. By (4.1, 4.2), the conductors of the ray class characters ψ and ψ' introduced in subsection 4.1 are given by,

$$\begin{aligned} F_\psi &= (2^a)_{\mathcal{O}_{\mathbf{K}}} = (2)_{\mathcal{O}_{\mathbf{K}}} = P_2^2 \quad \text{and} \\ F_{\psi'} &= (3\times 2^a)_{\mathcal{O}_{\mathbf{K}'}} = (6)_{\mathcal{O}_{\mathbf{K}'}} = 3P_2'^2, \end{aligned} \tag{5.30}$$

since $D'' = 3$ and so $\tilde{D}'' = 12$, and $a = 1$.

The calculation of ψ and ψ' is complicated by the fact that ideals in $\mathcal{O}_{\mathbf{K}}$ and $\mathcal{O}_{\mathbf{K}'}$ need not be principal ideals: $\mathcal{C}(\mathbf{K})$ is of order 4 and $\mathcal{C}(\mathbf{K}')$ is of order 2. In fact $[P_{11}]_{\mathcal{O}_{\mathbf{K}}}$ and $[P_{13}]_{\mathcal{O}_{\mathbf{K}}}$ are generators of $\mathcal{C}(\mathbf{K})$ so that, by (3.7), $[P_{11}]_{(2)}$ and $[P_{13}]_{(2)}$ together with $\mathcal{CP}_{(2)}(\mathbf{K})$ generate $\mathcal{C}_{(2)}(\mathbf{K})$. We find that $\psi = 1$ on $[P_{11}]_{(2)}$ and $[P_{13}]_{(2)}$ and on $[\gamma]_{(2)}$ if $\gamma \equiv 1 \bmod P_2$ and these values determine ψ.

Similarly, taking $[P_{13}']_{\mathcal{O}_{\mathbf{K}'}}$ as generator of $\mathcal{C}(\mathbf{K}')$, we eventually conclude that $\psi' = 1$ on $[P_{13}']_{(6)}$ and on $[\alpha]_{(6)}$, for $\alpha \in \mathcal{O}_{\mathbf{K}}\backslash(P_2 \cup 3\mathcal{O}_{\mathbf{K}'})$ if either both or neither of $\alpha \equiv 1 \bmod 2$ and $\alpha \equiv \pm 1$ or $\pm\sqrt{-10} \bmod 3$ are satisfied (this is the only way to ensure that the conductor is no larger than $6\mathcal{O}_{\mathbf{K}}$ — note that $\mathcal{O}_{\mathbf{K}'}/(3)^\times$ is cyclic of order 8 generated by the coset of $\mu' = 1+2\sqrt{-10}$.). Again these values determine ψ'.

they

We use the admissible pair (F, F') of conductors where $F = P_5P_3 4P_2$ and $F' = P_5'(3)4P_2'$. Thus $\mathcal{CP}_F(\mathbf{K})$ is the image of

$$(\mathcal{O}_{\mathbf{K}}/F)^\times \cong (\mathcal{O}_{\mathbf{K}}/P_5)^\times \times (\mathcal{O}_{\mathbf{K}}/P_3)^\times \times (\mathcal{O}_{\mathbf{K}}/4P_2)^\times, \tag{5.31}$$

(using (3.13) twice) and we denote its elements $[n, m, \alpha]_F$, accordingly, slightly generalising the notation introduced before (3.14). We find,

$$\ker\psi_F = \langle [P_{11}]_F, [P_{13}]_F, [n, m, 1+2\alpha]_F \mid n \in \mathbb{Z}-5\mathbb{Z}, m \in \mathbb{Z}-3\mathbb{Z}\rangle_{\mathrm{gp}}$$

(recalling (3.16) we see that we can take n and m to be integers). The classes $[n, m, 1+2\alpha]_F^{1-\gamma}$ turn out to be trivial, and for $\mu = 1 + 2\sqrt{-30}$, one has,

$$[P_{11}]_F^{1-\gamma} = [\mu/11]_F = [11\mu]_F = [1, -1, 3\mu]_F,$$

since $P_{11}^2 = \mu\mathcal{O}_{\mathbf{K}}$ and $11^2 \equiv 1 \bmod F$. Similarly,

$$[P_{13}]_F^{1-\gamma} = [(7 + 2\sqrt{-30})/13]_F = [-1, 1, 3\mu]_F.$$

So, in the notation of Theorem 4.3,

$$\begin{aligned} A_F &= (\ker \psi_F)^{1-\gamma} = \langle [1, -1, 3\mu)]_F, [-1, 1, 3\mu]_F \rangle_{\text{gp}} \\ &= \{[1]_F, [1, 1, -1]_F, [-1, 1, 3\mu]_F, [-1, 1, -3\mu]_F\} \end{aligned} \tag{5.32}$$

Also, $[1, 1, 1+\rho]_F^{1-\gamma} = [1, 1, (1+\rho)^2(31)^{-1}]_F = [1, 1, -3\mu]_F$, and

$$S_F = A_F[1, 1, -3\mu]_F = A_F[-1, 1, 1]_F. \tag{5.33}$$

Again $\ker \psi'_{F'} = B \cup B[1, \mu', 1 + \sqrt{-10}]_{F'}$, where

$$B = \langle [P'_{13}]_{F'}, [n, \beta, 1+2\alpha]_{F'} \mid n \in \mathbb{Z} - 5\mathbb{Z}, \beta \equiv \pm 1 \text{ or } \pm\sqrt{-10} \bmod 3 \rangle_{\text{gp}}.$$

We find $[P'_{13}]_{F'}^{1-\gamma} = [1, \sqrt{-10}, -1]_{F'}$, $[n, \beta, 1+2\alpha]_{F'}^{1-\gamma} = [1, \pm 1, 1]_{F'}$, and $[1, \mu', 1+\sqrt{-10}]_{F'}^{1-\gamma} = [1, -\sqrt{-10}, -3\mu']_{F'}$. So

$$\begin{aligned} A'_{F'} &= \langle [1, \sqrt{-10}, -1]_{F'}, [1, \sqrt{-10}, -3\mu']_{F'}, [1, -1, 1]_{F'} \rangle_{\text{gp}} \\ &= \{[1, \pm 1, 1]_{F'}, [1, \pm\sqrt{-10}, -1]_{F'}, [1, \pm 1, 3\mu']_{F'}, [1, \pm\sqrt{-10}, -3\mu']_{F'}\}. \end{aligned} \tag{5.34}$$

Also, $[1, 1, 1+\sqrt{-10}]_{F'}^{1-\gamma} = [1, 1, -3\mu']_{F'}$, and

$$S'_{F'} = A'_{F'}[1, 1, -3\mu']_{F'} = A'_{F'}[1, \pm 1, -1]_{F'}. \tag{5.35}$$

Now (because there are ideals of norm 13 in both $\mathcal{O}_{\mathbf{K}}$ and $\mathcal{O}_{\mathbf{K}'}$) there is an ideal $\tilde{P}$ in $\mathcal{O}_{\mathbf{KK}'}$ such that

$$\mathcal{N}_{\mathbf{KK}'/\mathbf{K}}(\tilde{P}) = P_{13} \quad \text{and} \quad \mathcal{N}_{\mathbf{KK}'/\mathbf{K}'}(\tilde{P}) = P'_{13}.$$

Again, let p be a prime such that $p \equiv 1 \bmod 12$. Then, by quadratic reciprocity,

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = 1.$$

So $\mathcal{O}_{\mathbf{K}''}$ has an ideal P''_p of norm p. It follows (from the identities of section 4.2) that

$$\mathcal{N}_{\mathbf{KK}'/\mathbf{K}}(P''_p\mathcal{O}_{\mathbf{KK}'}) = p\mathcal{O}_{\mathbf{K}} \quad \text{and} \quad \mathcal{N}_{\mathbf{KK}'/\mathbf{K}'}(P''_p\mathcal{O}_{\mathbf{KK}'}) = p\mathcal{O}_{\mathbf{K}'}.$$

Now, if n is prime to 5 and congruent to 1 mod 12 we may choose (by Dirichlet's theorem) $p \equiv n$ mod 120. Then, taking I of Theorem 4.3 to be $\tilde{P}P''_p$, we find that

$$[\mathcal{N}_{\mathbf{KK'/K}}(I)]_F = [pP_{13}]_F = [nP_{13}]_F \quad \text{and} \quad [\mathcal{N}_{\mathbf{KK'/K'}}(I)]_{F'} = [pP'_{13}]_{F'} = [nP'_{13}]_{F'}.$$

So that, by Theorem 4.3,

$$\theta([nP_{13}]_F(A_F - S_F); d) = \theta([nP'_{13}]_{F'}(A_{F'} - S_{F'}); d) \tag{5.36}$$

(Here sets A_F, S_F etc. stand for the sums of their elements.)

In particular, let $r \equiv \zeta$ mod 4, where $\zeta = \pm 1$; let $2 - s \equiv \zeta r$ mod 8; and let $t \equiv \zeta s \not\equiv 0$ mod 5. Then, choosing n congruent to s (and ζt) mod 5, to 1 mod 3 and to ζr mod 8, we have

$$\theta([P_{13}]_F[s, 1, 2-s]_F(A_F - S_F); d) = \theta([P'_{13}]_{F'}[t, 1, r]_{F'}(A_{F'} - S_{F'}); d), \tag{5.37}$$

where we have used the fact that $[\zeta, 1, \zeta]_{F'} = [1, \zeta, 1]_{F'}$ lies in $A'_{F'}$. Note that we can take

$$(s, r, t) = \begin{cases} (\quad 1, \quad 1, \quad 1\ ) & \ldots \quad \text{(i)} \\ (\ -11, \ -5, \quad 1\ ) & \ldots \quad \text{(ii)} \\ (\quad -3, \ -5, \ 13\ ) & \ldots \quad \text{(iii)} \\ (\quad -7, \quad 1, \ 13\ ) & \ldots \quad \text{(iv)} \end{cases} \tag{5.38}$$

We now set about reducing the identities (2.5) between Virasoro characters to identities like (5.37). As a first step, we rewrite the former using the V functions defined in 2.7. This gives,

$$\begin{aligned} \eta[V(1,4) \pm V(11,4)] &= [V(1,3) \pm V(5,3)][V(2,5) \mp V(8,5)] \\ \eta[V(-3,4) \pm V(7,4)] &= [V(1,3) \pm V(5,3)][V(-4,5) \mp V(14,5)] \end{aligned} \tag{5.39}$$

and

$$\begin{aligned} \eta V(2,4) &= V(2,3)[V(1,5) - V(19,5)] \\ \eta V(6,4) &= V(2,3)[V(7,5) - V(13,5)]. \end{aligned} \tag{5.40}$$

We concentrate on the four identities (5.39) here, since the last two (5.40) can be treated in a very similar way. Using the properties of V functions described in (2.8, 2.9), the relations (5.39) can be expressed in the following compact form,

$$V(1,2)V(s,4) = VV(r,t), \tag{5.41}$$

where

$$VV(r,t) \stackrel{\text{def}}{=} V(r,3)V(2t,5) + V(-5r,3)V(32t,5),$$

with (s, r, t) taking the values (5.38). These equations and hence the first four identities in (2.5) will follow from (5.37) when we show that, for the (s, r, t) of (5.38) and with $d = 240$,

$$2V(1,2)V(s,4) = \text{LHS of (5.37)} \tag{5.42}$$

and

$$2VV(r,t) = \text{RHS of (5.37)}. \tag{5.43}$$

We tackle (5.42) first. Applying (2.13) (both signs), we find

$$2\theta_{r,6}\theta_{s,20} = T(\alpha) \stackrel{\text{def}}{=} \theta(\alpha + J; d) + \theta(\bar{\alpha} + J; d),$$

where $\alpha = 10r + s\rho$, $J = 40P_3$ and $d = 2400$. Assume r prime to 6 and s to 10. We find that the h.c.f of $\alpha\mathcal{O}_{\mathbf{K}}$ and J is $H = P_2P_5$ and, since $F = JH^{-1}$,

$$T(\alpha) = \theta([\alpha H^{-1}]_F + [\bar{\alpha}H^{-1}]_F; 240) = \theta([\alpha H^{-1}]_F([1]_F + [\bar{\alpha}/\alpha]_F); 240).$$

Now put $\beta = 10 + \rho$ and, choosing b prime to 10, put $\hat{\alpha} = 10r + bs\rho$. Then

$$\alpha\hat{\beta} - \hat{\alpha}\beta = 10(b-1)(r-s)\rho \in 10F.$$

Since $H^2 = 10\mathcal{O}_{\mathbf{K}}$, it follows from Lemma 3.3(ii) that

$$[\hat{\alpha}/\alpha]_F = [\hat{\beta}/\beta]_F = \cdots = [b, 1, 2-b+(b-1)\rho]_F = \epsilon(b), \text{ say},$$

using the 3-component notation developed above. Also, $N(\beta H^{-1}) = 13$. So we may take $P_{13} = \beta H^{-1}$ and then $[\alpha H^{-1}]_F = [P_{13}]_F\epsilon(s)$. We have now

$$T(\alpha) = \theta([P_{13}]_F\epsilon(s)([1]_F + \epsilon(-1)); 240).$$

Now, since $-31 \equiv 2m+1 \bmod 2m(m+1)$ for $m = 2$ and $m = 4$, (2.7) can be rewritten for these m values as,

$$V(r,m) = \theta_{r,m(m+1)} - \theta_{31r,m(m+1)}.$$

So, taking $b = 31$ and $r = 1$ in α and $\hat{\alpha}$,

$$2V(1,2)V(s,4) = T(\alpha) + T(31\alpha) - T(\hat{\alpha}) - T(31\hat{\alpha}) = \theta(X(s); 240)$$

where

$$X(s) = [P_{13}]_F\epsilon(s)([1]_F + [31]_F)([1]_F + \epsilon(-1))([1]_F - \epsilon(31)).$$

Now, $[31]_F = [1,1,-1]_F$, $\epsilon(-1) = [-1,1,3\mu]_F$ and $\epsilon(31) = [1,1,3\mu] \in S_F$. So

$$X(s) = [P_{13}]_F\epsilon(s)(A_F - S_F) = [P_{13}]_F[s,1,2-s]_F(A_F - S_F),$$

if $s \equiv 1 \bmod 4$. Thus we have proved (5.42).

Applying (2.13) again, we find

$$2\theta_{r,12}\theta_{2t,30} = T(\alpha) \stackrel{\text{def}}{=} \theta(\alpha + J; d) + \theta(\bar{\alpha} + J; d)$$

where $\alpha = 5r + 2t\sqrt{-10}$, $J = 60P_2'$ and $d = 1200$.

We assume

$$r \equiv 1 \bmod 6 \text{ and } t \text{ prime to } 5 \text{ and } t \equiv 1 \bmod 3. \tag{5.44}$$

We find that the h.c.f of $\alpha\mathcal{O}_{\mathbf{K}}$ and J is $H = P_5'$ and, since $F' = JH^{-1}$,

$$T(\alpha) = \theta([\alpha H^{-1}]_{F'} + [\bar{\alpha} H^{-1}]_{F'}; 240) = \theta([\alpha H^{-1}]_{F'}([1]_{F'} + [\bar{\alpha}/\alpha]_{F'}); 240) \tag{5.45}$$

Now put $\beta = 5 + 2\sqrt{-10}$ and, choosing $a \equiv 1 \bmod 6$ and b prime to 15, put $\hat{\alpha} = 5ar + 2bt\sqrt{-10}$. Then

$$\alpha\hat{\beta} - \hat{\alpha}\beta = 10(b-a)(r-t)\sqrt{-10} \in 5F',$$

since $r \equiv t \bmod 3$. Since $H^2 = 5\mathcal{O}_{\mathbf{K}}$, it follows from Lemma 3.3(ii) that

$$[\hat{\alpha}/\alpha]_{F'} = [\hat{\beta}/\beta]_{F'} = \cdots = [1,1,a]_{F'}\epsilon'(b), \tag{5.46}$$

where

$$\begin{aligned} \epsilon'(b) &= [b, -(1+b) - (b-1)\sqrt{-10}, 1 + 2(b-1)\sqrt{-10}]_{F'} \\ &= [b,1,1]_{F'}, \quad \text{if } b \equiv 1 \bmod 6. \end{aligned}$$

In particular, from (5.45),

$$T(\alpha) = \theta([\alpha H^{-1}]_{F'}([1]_{F'} + \epsilon'(-1)); 240). \tag{5.47}$$

Now, from (2.7),

$$V(r,3) = \theta_{r,12} - \theta_{7r,12} \quad \text{and} \quad V(2t,5) = \theta_{2t,30} - \theta_{2(-11t),30}, \tag{5.48}$$

where we took the minus sign so that $7 \equiv -11 \equiv 1 \bmod 3$. This ensures that if $r = r_0$ and $t = t_0$ satisfy (5.44) then so do all the pairs (r,t) of the products $T(\alpha) = 2\theta_{r,12}\theta_{2t,30}$ in the expansion of $VV(r_0,t_0)$ using (5.48). Expressing each such product as in (5.47) and using (5.46) several times with different choices of a and b, we find, writing $\alpha_0 = 5r_0 + 2t_0\sqrt{-10}$, that

$$VV(r_0,t_0) = \theta(X; 240), \tag{5.49}$$

where

$$\begin{aligned} X = {} & [\alpha_0 H^{-1}]_{F'}([1]_{F'} + [1,1,-5]_{F'}\epsilon'(16)) \times \\ & \times ([1]_{F'} - \epsilon'(-11) - [1,1,7]_{F'} + [1,1,7]_{F'}\epsilon'(-11))([1]_{F'} + \epsilon'(-1)) \\ \ldots = {} & [\alpha_0 H^{-1}]_{F'}(A_{F'} - S_{F'}). \end{aligned}$$

Now, $N(\beta) = 25 + 40 = 65$. So $N(\beta H^{-1}) = 13$. Hence we may choose $P_{13}' = \beta H^{-1}$ and then

$$[\alpha_0 H^{-1}]_{F'} = [P_{13}']_{F'}[\alpha_0/\beta]_{F'} = [P_{13}']_{F'}[1,1,r_0]\epsilon(t_0).$$

So, if $t_0 \equiv 1 \bmod 6$,

$$X = [P_{13}']_{F'}[t_0,1,r_0]_{F'}(A_{F'} - S_{F'}).$$

Thus we have proved (5.43) and, as observed there, the first and second lines of (2.5) now follow.

6 Conclusions

Over the years, two-dimensional conformal field theory has proven to be a true goldmine for those studying string theory as well as statistical mechanics. Its underlying algebraic structure is the infinite dimensional Virasoro algebra. Although its representation theory has been thoroughly analysed, it is remarkable that identities between unitary minimal Virasoro characters of low level, of the kind discussed in this paper, have not been of use in any 'physical' context we are aware of. These identities could therefore be regarded as mathematical curiosities, but our aim here has been to provide a solid mathematical framework within which they naturally appear as the consequence of relations between two 'well chosen' imaginary quadratic extensions over $\mathbb{Q}$. The formalism used is borrowed from number theory, and provides, together with a new proof of the identities (2.4, 2.5), a new infinite family of identities between Virasoro characters at level 3 and level $m = 4a^2$, for a odd and $1 + 4a^2 = a'^2 p$ where p is prime.

From the number theory point of view, the interesting result is Theorem 4.3, which describes relations between ray class theta functions of two different imaginary quadratic fields $\mathbf{K}$ and $\mathbf{K}'$, under a certain number of constraints presented in Section 4. That these relations, when the pair ($\mathbf{K}$,$\mathbf{K}'$) is different from ($\mathbb{Q}[\sqrt{-2}], \mathbb{Q}[\sqrt{-1}]$) and ($\mathbb{Q}[\sqrt{-30}], \mathbb{Q}[\sqrt{-10}]$) but still obeys the constraints of Section 4, lead to other identities between unitary minimal Virasoro characters, is neither proven nor disproven at this stage.

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